UVAMT 2025 - Individual Round

I1. A two-digit positive integer x has the property that both $\sqrt{x}$ and $\sqrt[3]{x}$ are integers. Find x.
Solution: We need x to be a perfect square (since $\sqrt{x}$ is an integer) and a perfect cube (since $\sqrt[3]{x}$ is an integer). The two-digit perfect squares are 16, 25, 36, 49, 64, and 81; the two-digit perfect cubes are 27 and 64. The number 64 is the only one that appears on both lists, so the answer is **64**.

I2. A rectangle has perimeter 70 and area 300. How long is its diagonal?

Solution: Suppose the rectangle has width *w* and height *h*. We want the length of the diagonal *d*: by the Pythagorean Theorem, we have $d^2 = w^2 + h^2$, so $d = \sqrt{w^2 + h^2}$.

Since the perimeter is 70, we know that:
$2w + 2h = 70$
$\implies w + h = 35$
$\implies (w + h)^2 = 35^2$
$\implies w^2 + 2wh + h^2 = 1225$

Since the area is 300, we also know that:
$wh = 300$
$\implies w^2 + 2(300) + h^2 = 1225$
$\implies w^2 + h^2 = 1225 - 2(300)$
$\implies w^2 + h^2 = 625$
$\implies d = \sqrt{w^2 + h^2} = \sqrt{625} = 25.$

Therefore, the length of the diagonal *d* is **25.**

I3. Utkarsh comes across some snowballs in a field. Every minute, Utkarsh choose three snowballs and combines them into one. Suppose that after 20 minutes, there are 25 separate snowballs left in the field. How many snowballs must there have been at the start?

Solution: After every minute, 3 snowballs get replaced by 1, so the number of snowballs goes down by 2 every minute. So after 20 minutes, the number of snowballs has decreased by 2(20) = 40. Since 25 snowballs remain at the end, that means the initial number of snowballs must have been 25 + 40 = **65**.

I4. Utkarsh and Vincent are running laps around a long circular track with circumference 21 miles. Utkarsh runs at a constant pace of 6 minutes per mile, while Vincent runs at a constant pace of 8 minutes per mile in the opposite direction. If they started at the same point, how many miles did Vincent run before passing Utkarsh for the first time?

Solution: Suppose that it took m minutes for Vincent to pass Utkarsh for the first time.

In those m minutes, Vincent ran $m/8$ miles (since he takes 8 minutes to run 1 mile), and Utkarsh ran $m/6$ miles (since he takes 6 minutes to run 1 mile). Also, at the instant that they first meet again, their combined distance traveled equals the circumference of the track, 21 miles. Then:

$$\frac{m}{8} + \frac{m}{6} = 21$$

$$\implies \frac{3m}{24} + \frac{4m}{24} = 21$$

$$\implies \frac{7m}{24} = 21$$

$$\implies 7m = 21(24) = 504$$

$$\implies m = \frac{504}{7} = 72$$

The total distance that Vincent ran is therefore $m/8 = 72/8 =$ **9** miles.

I5. Suppose there are 31 people signed up for UVAMT. Teams have a maximum size of 6, but any set of teams containing a total of at most 6 people can be combined together. However, teams cannot be broken up. Find the maximum possible number of teams (totaling 31 people) such that it is not possible to combine any of them.

Solution: Eight teams are possible, with team sizes of 4, 4, 4, 4, 4, 4, 4, 3. To show that nine teams aren't possible, notice that most one team can have a size of 3 or less (otherwise you could combine them). So if nine teams were possible, at least eight of them would need to have 4 or more members. This is a total of $4(8) = 32$ people, which is too many - this means nine is impossible, so the maximum possible amount of teams is **8**.

I6. Vincent has 6 buckets, with capacities 2, 3, 4, 5, 6, and 7 pounds. He also has 6 bricks, weighing 1, 2, 3, 4, 5, and 6 pounds. He wants to put exactly one brick in each bucket such that no bucket is above capacity. How many ways are there for him to accomplish this?

Solution: First, place the 6-pound brick: there are two possible buckets to put it in (the buckets with capacity 6 and 7 pounds). Next, place the 5-pound brick: there are still two possible buckets (the buckets with capacities 5, 6, and 7 pounds all work, but one of them is already occupied by the previous brick). Similarly, there are two possible buckets for the 4-, 3-, and 2-pound bricks, if you place them in that order. Finally, the 1-pound brick only has one remaining bucket to go in. The total number of possibilities is $2\times2\times2\times2\times2\times1 = \mathbf{32}$.

I7. In Chickenville, each household has either 1, 2, or 3 children. 20% of the children are only children, but 35% of households have only one child. Determine the average number of children per household.

Solution: Setting up a system of equations will work. But there's a better way with dimensional analysis:

$$\text{Average Children Per Household} = \frac{\text{Total Children}}{\text{Total Households}}$$

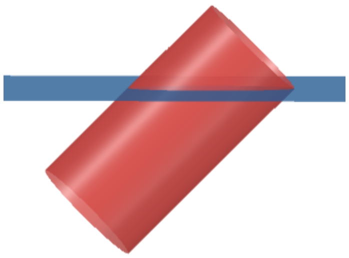
$$= \frac{\begin{array}{c}\text{Number Of One-}\\\text{Child Households}\end{array}}{\begin{array}{c}\text{Total}\\\text{Households}\end{array}} \times \frac{\begin{array}{c}\text{Number Of}\\\text{Only Children}\end{array}}{\begin{array}{c}\text{Number Of One-}\\\text{Child Households}\end{array}} \times \frac{\begin{array}{c}\text{Total}\\\text{Children}\end{array}}{\begin{array}{c}\text{Number Of}\\\text{Only Children}\end{array}}$$

$$= \frac{\begin{array}{c}\text{Number Of One-}\\\text{Child Households}\end{array}}{\begin{array}{c}\text{Total}\\\text{Households}\end{array}} \times \frac{\begin{array}{c}\text{Number Of}\\\text{Only Children}\end{array}}{\begin{array}{c}\text{Number Of One-}\\\text{Child Households}\end{array}} \div \frac{\begin{array}{c}\text{Number Of}\\\text{Only Children}\end{array}}{\begin{array}{c}\text{Total}\\\text{Children}\end{array}}$$

$$= 0.35 \times 1 \div 0.2$$

$$= \mathbf{1.75.}$$

I8. A cylindrical container with radius 1 and height 4 is partially filled with water. If tilted 45 degrees from the upright position, no water will spill, but if tilted any more than that, water will start to spill. Determine the volume of water in the cup.



The key to this problem is having a good diagram, as shown above. The bottom half of the cylinder is completely below the water level, contributing a volume of $\pi r^2 h = \pi(1)^2(2) = 2\pi$. If we only look at the top half of the cylinder, exactly half of this portion is above the water level, since the part above the water level forms the same 3D shape as the part below. So this part contributes half its volume: $\frac{\pi r^2 h}{2} = \frac{\pi(1)^2(2)}{2} = \frac{2\pi}{2} = \pi$. So the total volume is $2\pi + \pi = \mathbf{3\pi}$.

I9. A random number x is selected uniformly in $[0, 1]$. Mikhail can then apply as many moves to x as he wants, where a move is defined as rounding x to the decimal position of his choice. Mikhail wins if he can set x to 1 in finitely many moves. Find Mikhail's probability of winning.

Solution: Mikhail can win if and only if x is greater than 4/9. If it is greater than 4/9, then the first decimal digit of x excluding 4s is at least 5. Using this fact, the optimal strategy for Mikhail straightforward, as shown below:

$0.44462 \to 0.445 \to 0.45 \to 0.5 \to 1$

If x is less than or equal to 4/9, the first decimal digit of x excluding 4s is at most 3. Because of this, however Mikhail proceeds, he will eventually get stuck, as shown below:

$0.444348 \to 0.44435 \to 0.4444 \to$ All subsequent moves will only decrease the number!
$0.4419 \to 0.442 \to$ All subsequent moves will only decrease the number!

Therefore, he wins whenever x is between 4/9 and 1, which happens with probability **5/9**.

I10. A rectangular prism with positive integer side lengths has volume V and surface area $2V$. What is the sum of all possible values of V?

Solution: If the rectangle has positive integer side lengths a, b, c, then we need to solve the following equations:

Volume = $V = abc$
Surface Area = $2V = 2(ab + ac + bc) \implies V = ab + ac + bc$

Setting these equal to each other, we have that $abc = ab + ac + bc$.

Now, we'll divide both sides by abc to get $1 = 1/a + 1/b + 1/c$.

Clearly, a, b, and c all have to be greater than 1 - otherwise, the right side would be too big.

If a, b, and c are all at least 3, then the only solution is $a = 3$, $b = 3$, $c = 3$. Setting any of the variables higher would make the right side too small.

Otherwise, one of a, b, and c is less than 3. Since we know they all have to be greater than 1, one of them has to equal 2. Assume without loss of generality that a = 2 and $b \leq c$.

- If $b = 2$, then the right side becomes too big.
- If $b = 3$, then $c = 6$.
- If $b = 4$, then $c = 4$.
- If b $\geq$ 5, then since we assumed $b \leq c$, the right side would become too small.

Therefore, the only solutions up to a permutation of the side lengths are (a, b, c) = (3, 3, 3), (2, 3, 6), and (2, 4, 4). Therefore, the possible values of $V = abc$ are:

- If (a, b, c) = (3, 3, 3), then $V = 3\times3\times3 = 27$.
- If (a, b, c) = (2, 3, 6), then $V = 2\times3\times6 = 36$.
- If (a, b, c) = (2, 4, 4), then $V = 2\times4\times4 = 32$.

Therefore, the sum of all possible values of V is 27+36+32 = **95**.

I11. Suppose that sets $S_1, S_2, \cdots, S_{12}$ satisfy the following properties:

- Any three of these sets have exactly one element in common.
- Any four of these sets have no elements in common.

Find the minimum possible value of $|S_1|$.

Solution: For each possible combination of three sets, we need to add a unique element into those sets. The condition that any four of these sets have no elements in common makes it so that this element must be unique - it can't be reused for any other combination of three sets. Using this construction, the number of elements in S_1 is just the number of ways to choose three sets among $\mathrm{S}_1, \mathrm{S}_2, \cdots, \mathrm{S}_{12}$ such that S_1 is chosen. Since S_1 must be chosen, we have to choose two other sets among the remaining sets $S_2, \cdots, S_{12}$, so there are $\binom{11}{2} = 55$ elements in S_1. All of these elements were forced to be added (there is no way we could have avoided adding them), so the minimum possible size of S_1 is **55**.

Note: Larger sets satisfying the constraints are possible, by adding single elements to S_1 not present in any other set. However, 55 is still minimal.

I12. Let S be a set of integers. Suppose that for any two distinct integers $x, y \in S$ and any nonnegative integer k, we have $2^k x \not\equiv y \pmod{2^{12}-1}$. Find the maximum possible size of S.

Solution: Without loss of generality, assume that S only contains numbers between 0 (inclusive) and $2^{12}-1$ (exclusive). Express x as a number in binary of length 12, left-padding with zeros as needed. Then the operation of multiplying it by 2 and reducing mod $2^{12}-1$ is equivalent to cycling all the bits left by one position: each bit moves left by one position; if a 1 bit moves into the 2^{12}-position, then that 1 goes to the end of the number (because $2^{12} \equiv 2^0 \bmod 2^{12}-1$). Then multiplying x by 2^k is equivalent to doing this operation k times; that is, cycling all the bits left by k positions. Therefore, if two numbers are cyclic shifts of each other when written as length-12 binary numbers (that is, you can reach one number from the other by cycling the bits by some number of positions), they cannot both be included. On the other hand, if two numbers x, y between 0 (inclusive) and $2^{12}-1$ (exclusive) are not cyclic shifts of each other, there cannot exist an integer k such that $2^k x \equiv y \pmod{2^{12}-1}$. Therefore, the answer is just the number of length-12 binary strings with cyclic shifts considered equivalent, minus 1 because the all-ones binary string is not in the range 0 (inclusive) to $2^{12}-1$ (exclusive).

This quantity can be computed using Burnside's Lemma.

For each rotation k (from 0 to 11), we count how many 12-bit strings stay the same when their bits are rotated left by k positions. Burnside's Lemma tells us that the average of these values equals the quantity we're looking for.

The number of such fixed strings depends on the greatest common divisor of 12 and k. Specifically, the number of 12-bit strings fixed under rotation by k positions is $2^{\gcd(12,k)}$. This is because the string must repeat every $\gcd(12, k)$ positions to be unchanged by rotation.

Let's compute this sum:

- $k = 0: \ 2^{\gcd(12,k)} = 2^{12} = 4096$

- $k = 1: 2^{\gcd(12,k)} = 2^1 = 2$
- $k = 2: 2^{\gcd(12,k)} = 2^2 = 4$
- $k = 3: 2^{\gcd(12,k)} = 2^3 = 8$
- $k = 4: 2^{\gcd(12,k)} = 2^4 = 16$
- $k = 5: 2^{\gcd(12,k)} = 2^1 = 2$
- $k = 6: 2^{\gcd(12,k)} = 2^6 = 64$
- $k = 7: 2^{\gcd(12,k)} = 2^1 = 2$
- $k = 8: 2^{\gcd(12,k)} = 2^4 = 16$
- $k = 9: 2^{\gcd(12,k)} = 2^3 = 8$
- $k = 10: 2^{\gcd(12,k)} = 2^2 = 4$
- $k = 11: 2^{\gcd(12,k)} = 2^1 = 2$

Now add these up and divide by 12: (4096+2+4+8+16+2+64+2+16+8+4+2)/12 = 352.

Finally, we subtract 1, because we must exclude the class containing the all-ones string (111111111111), since it corresponds to the number $2^{12} - 1 = 4095$, which is out of our range. Therefore, the maximum possible size of S is **351**.

UVAMT 2025 - Team Round

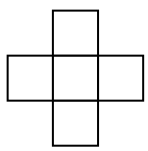
T1. On a table, some coins are placed in a row, each showing heads or tails, such that:

- Exactly 3 coins are showing tails,
- Each tail is adjacent to two heads, and
- Each head is adjacent to exactly one other head.

How many coins must there be on the table in total?

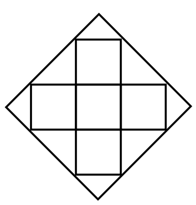
Solution: In a valid arrangement, each of the tails must be sandwiched between pairs of adjacent heads. This gives us the arrangement H H T H H T H H T H H, which contains a total of **11** coins.

T2. Remove four 1x1 squares from the corners of a 3x3 square to obtain the following figure:

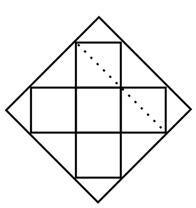


It turns out that this figure fits in a square smaller than 3x3! What's the minimum side length of a square that this figure can fit in?

Solution: To achieve this smaller fit, use a square rotated by 45 degrees:



In the figure below, notice that the dotted line has a length of twice the diagonal of one of the smaller squares, and has the same length as a side of the large square:



The length of the diagonal in a square of side length 1 is $\sqrt{1^2 + 1^2} = \sqrt{2}$, by the Pythagorean theorem. Therefore, the side length of the large square is $\mathbf{2\sqrt{2}}$.

T3. Mikhail wants to fill in the empty cells in this grid with integers, such that every row, column, and 3-cell diagonal sums to the same value v. Find v.

		4
3		
	5	8

Solution: We'll assign variables to the unknown cells for extra clarity:

(a)	*(b)*	4
3	*(c)*	*(d)*
(e)	5	8

From the middle row and rightmost column, we have $3+c+d = 4+d+8$, so cancelling d from both sides we get $3+c = 4+8$. Therefore, $c = 9$.

From the bottom row and leftmost column, we have $e+5+8 = a+3+e$, so cancelling e from both sides we get $5+8 = a+3$. Therefore, $a = 10$.

Finally, from the diagonal containing a, c, and 8, we get $v = a+c+8$, which equals 10+9+8=**27**.

Note: from here, it's relatively straightforward to fill in the grid, even if not strictly necessary to solve the problem:

10	13	4
3	9	15
14	5	8

As we wanted, all rows, columns, and diagonals in the completed grid add up to **27**.

T4. How many 4-digit numbers have exactly 3 odd digits?

Solution: We will split into cases, indicating which position contains the one even digit.

- If the first digit is even, then there are 4 choices for it (2, 4, 6, or 8), and 5 choices for

each of the other three odd digits (1, 3, 5, 7, or 9). Note that the first digit cannot be zero. This gives a total of $4\times5\times5\times5 = 500$ solutions.
- If the second digit is even, then there are 5 choices for it (0, 2, 4, 6, or 8), and 5 choices for each of the other three odd digits (1, 3, 5, 7, or 9). (The difference between this case and the previous one is that non-leading digits are allowed to be zero.) This gives a total of $5\times5\times5\times5 = 625$ solutions.
- If the third digit is even, we get another 625 solutions, using the same logic as the previous case.
- Finally, if the last digit is the even one, we get another 625 solutions.

The total number of solutions is therefore 500+625+625+625 = **2375**.

T5. Suppose that x is an integer such that both x and $x + 45$ are perfect squares. Determine the sum of all possible values of x.

Solution: We will set $x = a^2$ and $x + 45 = b^2$. Then:

$$b^2 - a^2 = (x + 45) - x = 45$$
$$\implies (b + a)(b - a) = 45$$

The possible integer factor pairs $(b + a,\ b - a)$ of 45 are (45, 1), (15, 3), and (9, 5).

- If $(b + a,\ b - a) = (45,\ 1)$, then:
  $(b + a) - (b - a) = 44 \implies 2a = 44 \implies a = 22 \implies a^2 = 484 \implies x = 484.$
- If $(b + a,\ b - a) = (15,\ 3)$, then:
  $(b + a) - (b - a) = 12 \implies 2a = 12 \implies a = 6 \implies a^2 = 36 \implies x = 36.$
- If $(b + a,\ b - a) = (9,\ 5)$, then:
  $(b + a) - (b - a) = 4 \implies 2a = 4 \implies a = 2 \implies a^2 = 4 \implies x = 4.$

The sum of all possible values of x is 484 + 36 + 4 = **524**.

(Technical note: you can get more possible factor pairs if you swap d and e, or negate them both, but if you follow the same process with these solutions, you won't get any new possible values of x.)

T6. Thirty students are in a school that offers three courses. Each course is taken by exactly 21 students. Find the maximum possible number of students taking exactly one course.

Solution: Assign the following variables.

- x = number of students taking exactly one course
- y = number of students taking exactly two courses
- z = number of students taking exactly three courses.

Since there are 30 students, we know that $x + y + z \leq 30$. Rearrange this to get $y \leq 30 - x - z$.

Each course has 21 students, so total enrollment is $21 \times 3 = 63$. So we can also deduce that $x + 2y + 3z = 63$.

Substituting, we get:
$$\begin{aligned} 63 &= x + 2y + 3z \\ &\leq x + 2(30 - x - z) + 3z \\ &= 60 - x + z. \end{aligned}$$

Therefore:
$$\begin{aligned} &63 \leq 60 - x + z \\ &\implies 3 \leq -x + z \\ &\implies 3 + x \leq z \\ &\implies z \geq 3 + x. \end{aligned}$$

If x were at least 14, then z would have to be at least 17, giving a total of 31 students, which is too many. Therefore, x has to be at most 13.

To see that this is possible to achieve, we use the following allocation:

- 4 students taking only course A
- 4 students taking only course B
- 5 students taking only course C
- 1 student taking courses A and B
- 16 students taking courses A, B, and C

This arrangement totals 30 students, and exactly 21 students are taking each course. A total of 13 students are taking only one course, so the maximum value of 13 is achievable. Since it is impossible to do better, the maximum possible number is **13**.

T7. Let a positive integer n be *satisfying* if there exists a geometric progression $a,\ b,\ c$ of positive integers $a < b < c$ with $a + b + c = n$. Find the smallest n such that both n and $n + 2$ are satisfying.

Solution: We'll take cases on the common ratio *r*. Note that since $a < b < c$, this common ratio must be (strictly) greater than 1.

- r = 2: the geometric progression is of the form x, 2x, 4x, for some positive integer x. Their sum is 7x, so the numbers 7, 14, 21, 28, ... are all satisfying.
- r = 3: the geometric progression is of the form x, 3x, 9x, for some positive integer x. Their sum is 13x, so the numbers 13, 26, 39, ... are all satisfying.
- r = 4: the geometric progression is of the form x, 4x, 16x, for some positive integer x. Their sum is 21x, so the numbers 21, 42, 63, ... are all satisfying.
- r = 3/2: the geometric progression is of the form 4x, 6x, 9x, for some positive integer x. Their sum is 19x, so the numbers 19, 38, 57 ... are all satisfying.

From these lists, we see that 19 and 21 are both satisfying, so $n = 19$ satisfies the conditions of the problem (both n and n+2 are satisfying). It's not hard to see that any larger or more complex value of r won't give you any satisfying numbers smaller than 19. So, the value of n we found is optimal, so the answer is **19**.

T8. Find the sum of all real numbers $a < 2025$ such that $x\lfloor x \rfloor = a$ has at least two solutions.

Solution: First, notice that the function $x\lfloor x \rfloor$ is increasing for $x > 0$, and decreasing for $x < 0$.

We consider the range of $x\lfloor x \rfloor$ when $x > 0$. When x is an integer, $x\lfloor x \rfloor$ is a perfect square. But if x decreases by an arbitrarily small amount to x', then $x'\lfloor x' \rfloor$ becomes arbitrarily close to $x(x-1)$. So, there's a discontinuous jump in the values that the function attained from $x(x-1)$ to x^2, for all positive integer x. Note that the value $x(x-1)$ is never actually attained, though the value x^2 is. Therefore, using this fact about where the discontinuous jumps are, and noting that the function is continuous everywhere else, the range of $x\lfloor x \rfloor$ when $x > 0$ is $0 \cup [1,2) \cup [4,6) \cup [9,12) \cup [16,20) \cdots$.

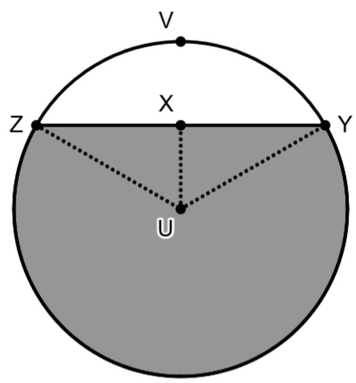
Next, we consider the range of $x\lfloor x \rfloor$ when $x < 0$. By the same logic as before, there's a discontinuous jump in the values that the function attained from $x(x-1)$ to x^2. But now, the values attained are different: using this fact about where the discontinuous jumps are, the range of $x\lfloor x \rfloor$ when $x < 0$ is $\cdots [25,20) \cup [16,12) \cup [9,6) \cup [4,2) \cup [1,0)$.

For any nonzero value of a, the equation $x\lfloor x \rfloor = a$ has at most one solution for which x > 0, and at most one solution for which x < 0 (this is because the function $x\lfloor x \rfloor$ is *strictly* increasing/decreasing over these intervals when a is nonzero), so this equation has two solutions if and only if a is in both of the ranges listed above - the range $x\lfloor x \rfloor$ when $x > 0$ and when x < 0. These values of a are just the perfect squares. We can safely ignore the $a = 0$ case because it won't affect the sum. Therefore, the sum we want is just all the perfect squares below 2025:

namely, $1^2 + 2^2 + 3^2 + \cdots + 44^2$. This is equal to $\frac{44(44+1)(2(44)+1)}{6}$ = **29370**.

T9. Utkarsh and Vincent are moving to UVAMT grounds, which are shaped like a perfect circle. Utkarsh is at the center of grounds, and Vincent is at the edge. They pick a random spot uniformly within grounds to meet up. What is the probability that Utkarsh is closer to this spot than Vincent?

Solution: We will use the following diagram, where Utkarsh is located at point U, and Vincent at point V.



The set of points equidistant from U and V is the perpendicular bisector of segment UV. As such, we define line ZY to be the perpendicular bisector of segment UV.

Without loss of generality, let the radius of this circle be 1 unit. Then the shaded area divides into a circular sector and a triangle ZUY.

First, we will find the area of triangle ZUY. Its height UX is 1/2, since UV = 1 and X is the midpoint of U and V (because ZY was defined as the perpendicular bisector). To find the base, notice that UY = 1 and UX = 1/2, so triangle XUY is a 30-60-90 triangle, so XY = $\sqrt{3}/2$. Since X is the midpoint of ZY, we have ZY = 2XY = $\sqrt{3}$. Therefore, the area of triangle ZUY is $(1/2)\left(\sqrt{3}\right)(1/2) = \sqrt{3}/4$.

Now we will find the area of the circular sector. From before, we had that triangle XUY is a 30-60-90 triangle. So ∠ZUX = ∠XUY = 60°, so the circular sector occupies the remaining 360° - 60° - 60° = 240°, or 2/3 of the circle. Thus the area of this circular sector is $(2/3)\pi$.

Since the random spot is picked uniformly randomly inside the circle, the probability we want is just the area of the shaded region (denoting all points closer to U than to V) divided by the area of the circle (π). This is equal to $\frac{(2/3)\pi + \sqrt{3}/4}{\pi} = \mathbf{\frac{2}{3} + \frac{\sqrt{3}}{4\pi}}$.

T10. How many non-constant polynomials with integer coefficients pass through the point $(-2,\ 27)$ and have all positive integer roots?

Solution: Let $p(x) = a(x - r_1)(x - r_2)\ \cdots\ (x - r_d)$ be a degree *d* polynomial that satisfies the constraints of the problem. We need $p(-2) = a(-2 - r_1)(-2 - r_2)\ \cdots\ (-2 - r_d) = 27$. Therefore, the product $(-2 - r_1)(-2 - r_2)\ \cdots\ (-2 - r_d)$ must be a factor of 27, up to sign. Notice that each of the linear factors $(-2 - r_1), (-2 - r_2),\ \ldots$ must be -3 or less, which will greatly restrict how many factors we can have. We will take cases on the degree *d*.

Case 1 (d=1): We have one linear factor that could evaluate to -3, -9, or -27. This gives us the following polynomials:

- $p(x) = -9(x - 1)$: linear factor evaluates to -3
- $p(x) = -3(x - 7)$: linear factor evaluates to -9
- $p(x) = -(x - 25)$: linear factor evaluates to -27

Case 2 (d=2): We need two linear factors; they could evaluate to (-3)(-3) or (-3)(-9). This gives us the following polynomials:

- $p(x) = 3(x - 1)(x - 1)$: linear factors evaluate to (-3)(-3) = 9
- $p(x) = (x - 1)(x - 7)$: linear factors evaluate to (-3)(-9) = 27

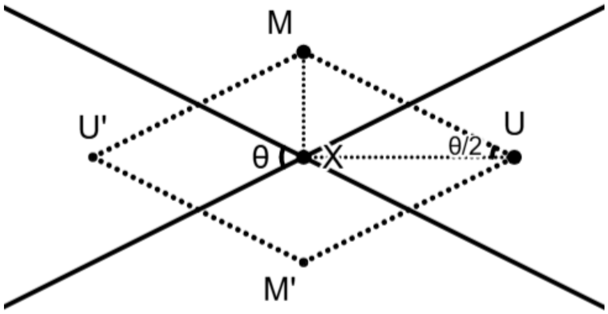
Case 3 (d=3): We need three linear factors; they must all evaluate to -3 (otherwise the product would exceed 27 in absolute value). This gives us the following polynomials:

- $p(x) = -(x - 1)(x - 1)(x - 1)$: linear factors evaluate to (-3)(-3)(-3) = -27

If the degree exceeds 3, then the product would get too large (its absolute value would be at least $3^4 = 81$, which is greater than 27. So, this covers all cases, giving a total of **6** polynomials.

T11. Two straight roads intersect at an angle θ. Utkarsh and Mikhail are both 1 mile away from each of the roads, but Utkarsh is twice as far from their intersection as Mikhail is. Find $\cos\theta$. Note: You may assume that the roads are infinitely long in both directions.

Solution: Consider the following diagram, where U represents Utkarsh's position, M represents Mikhail's position, and the two intersecting roads are denoted as the solid lines.



We reflect U across X to obtain the point U', and reflect M across X to obtain the point M'. Since Each of M, U, M', U' are one mile away from the roads, so each side MU, UM', M'U', U'M measures exactly two miles: two of these one mile distances joined together. Thus MUM'U' is a rhombus. Diagonals of a rhombus are perpendicular, so ∠MXU is a right angle.

Since the dotted lines are parallel with their corresponding solid lines (since the two points they're defined by are equidistant from the road), we have that ∠MUM' = θ. Thus ∠MUX = $\theta/2$.

Since Utkarsh is twice as far from their intersection as Mikhail is, we have UX = 2MX. Thus UM = $\sqrt{5}$MX by the Pythagorean Theorem. Thus
$\cos(\theta/2) = UX/UM = 2MX/\sqrt{5}MX = 2/\sqrt{5}$, so
$\cos\theta = 2\cos^2(\theta/2) - 1 = 2\left(2/\sqrt{5}\right)^2 - 1 = 2(4/5) - 1 = \mathbf{3/5}$.

Note: θ could also be taken as the other possible angle of intersection between the two roads; $\mathbf{-3/5}$ was also an acceptable answer.

T12. Eight vertices are placed in 3D space to form a unit cube. Utkarsh chooses four distinct vertices randomly out of these eight. What is the expected volume of the tetrahedron formed by these four vertices? (Note: if all four vertices lie on the same plane, the volume is said to be 0 by convention.)

Solution: There are $\binom{8}{4} = 70$ distinct ways to choose four vertices out of 8, and each of these choices are equally likely. We will take cases on the type of tetrahedron formed:

Case 1: All four vertices lie on the same plane. Then either they lie on the same face (6 cases), or they are formed by taking the vertices among two pairs of opposite edges (6 cases, because the 12 edges of the cube form 6 pairs). This gives 12 cases total, each with a volume of 0.

Case 2: The four vertices form a regular tetrahedron. This happens when you choose alternating vertices - that is, a set of vertices where no two of them share an edge. The only such possibilities are $\{(0,0,0),(0,1,1),(1,1,0),(1,0,1)\}$ and $\{(0,0,1),(0,1,0),(1,0,0),(1,1,1)\}$. This gives 2 cases total. The volume of a regular tetrahedron with side length x is $x^3 / \left(6\sqrt{2}\right)$, and the side length of these tetrahedra are $\sqrt{2}$, giving a volume of $\frac{\sqrt{2}^3}{6\sqrt{2}} = \frac{2\sqrt{2}}{6\sqrt{2}} = \frac{1}{3}$. (Alternatively, these regular tetrahedra can be thought of as taking a cube and removing four right-angled tetrahedra from the corners not selected; each of these right-angled tetrahedra has base area 1/2 and height 1, giving a volume of (1/2)(1)/3 = 1/6, so the remaining volume is 1 - 4(1/6) = 1/3.)

Case 3: The remaining 70 - 12 - 2 = 56 cases all contain one face with three selected vertices (the last selected vertex is on the opposite face). If we consider the face with three selected vertices to form the base of our tetrahedron, the base has area 1/2, and the height of the tetrahedron is 1, so its volume is (1/2)(1)/3 = 1/6.

Therefore, the expected volume is (12/70)(0) + (2/70)(1/3) + (56/70)(1/6) = 1/7.

T13. Utkarsh selects an integer between 1 and n inclusive, and Vincent makes a series of guesses to try to determine Utkarsh's number. When he makes a guess, Utkarsh tells him whether it was greater than, less than, or equal to the correct number. If it was greater, Utkarsh gets 3 points; if it was less, Utkarsh gets 1 point; if it was correct, the game ends. Find the greatest n such that Vincent can ensure Utkarsh scores at most 10 points.

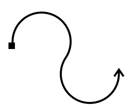
Solution: We use recursion. Let f(x) be the greatest *n* such that Vincent can ensure Utkarsh scores at most *x* points. It's clear from testing small cases that f(0) = 1, f(1) = 2, and f(2) = 3. For x at least 3, Vincent's optimal strategy is to make a guess such that if it was greater than the correct number, f(x-3) possible numbers remain (since he would have 3 fewer points to spare in this case), and if it was less than the correct number, f(x-1) possible numbers remain (since he would have 1 fewer point to spare in this case). Therefore, the function f(x) satisfies the recurrence f(x) = f(x-3) + f(x-1) + 1, where the +1 represents the case where Vincent guesses correctly. Calculating out this recurrence, we get:

- f(3) = f(0) + f(2) + 1 = 1 + 3 + 1 = 5
- f(4) = f(1) + f(3) + 1 = 2 + 5 + 1 = 8
- f(5) + f(2) + f(4) + 1 = 3 + 8 + 1 = 12
- f(6) = f(3) + f(5) + 1 = 5 + 12 + 1 = 18
- f(7) = f(4) + f(6) + 1 = 8 + 18 + 1 = 27

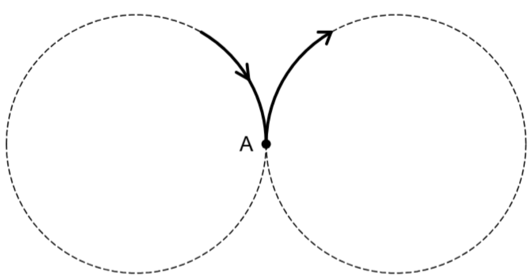
- f(8) = f(5) + f(7) + 1 = 12 + 27 + 1 = 40
- f(9) = f(6) + f(8) + 1 = 18 + 40 + 1 = 59
- f(10) = f(7) + f(9) + 1 = 27 + 59 + 1 = 87

Therefore, the greatest *n* such that Vincent can ensure Utkarsh scores at most 10 points is f(10) = **87**.

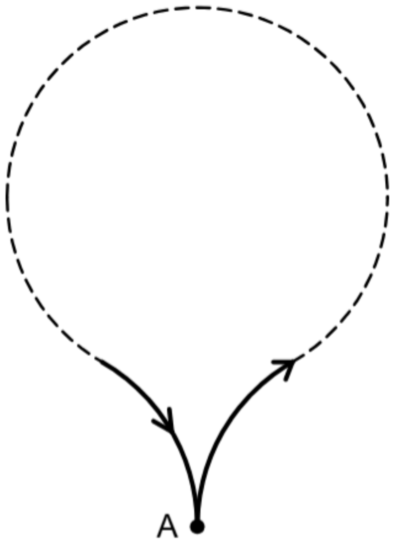
T14. You're driving a go-kart without gas or brakes - just a steering wheel. The go-kart travels in arcs of radius 1, and you can instantaneously change the direction of its arc between clockwise and counterclockwise, but this is all the control you have. You're currently at point A traveling due north. What's the minimum distance you must travel to return to point A traveling due south? (An example of the go-kart's movement pattern is given below.)



Solution: Although it's possible to go straight by switching between clockwise and counterclockwise arbitrarily quickly, it's not actually optimal to do that, because (on an intuitive level) the distance traveled while going straight could have been used to make progress. So, an optimal path should start and end as follows:



Then, to connect these two partial paths together, the most efficient way is to loop around the top, as follows:



The initial and final portions (denoted by solid lines) should be the same length, so that they can connect together. The only free parameter in this setup is how long the initial/final portions should be. Suppose that, during the initial portion, the go-kart has turned θ degrees clockwise. Then, in the final portion, the go-kart also turns θ degrees clockwise. The total net rotation needs to be 180 degrees counterclockwise, so the portion in the dotted lines should be a turn of $180 + 2\theta$ degrees counterclockwise.

We need the x-coordinates to match up: that is, we want to end at A, not at some point to the left or right of A. An angle of $\theta = 60°$ works: the initial and final portions both result in a net x-coordinate change of +0.5 units, and the 300-degree portion results in a net x-coordinate change of -1 unit.

Therefore, the total path length is $\pi/3$ (initial portion) + $5\pi/3$ (300-degree portion) + $\pi/3$ (final portion) = $\mathbf{7\pi/3}$.

T15. For positive integer x, let $f(x) = \begin{cases} x/2 & (x \text{ even}) \\ x+1 & (x \text{ odd}) \end{cases}$. Let $g(x)$ be the minimum k such that $\underbrace{f(f(\ldots f(x)))}_{k} = 1$. Compute $\sum_{x=2}^{2^{2025}} g(x)$ mod 999.

Solution: Let $h(n) = \sum_{x=2^n+1}^{2^{n+1}} g(x)$. We will derive a recurrence relation for h(n).

Note that $g(x) = 1 + g(x/2)$ if x is even, and $g(x) = 1 + g(x+1)$ if x is odd.

Split $h(n)$ into its odd and even terms, and simplify as follows:

$$h(n) = \sum_{x=2^n+1\ odd}^{2^{n+1}} g(x) + \sum_{x=2^n+1\ even}^{2^{n+1}} g(x)$$

$$= \sum_{x=2^n+1\ odd}^{2^{n+1}} (1 + g(x+1)) + \sum_{x=2^n+1\ even}^{2^{n+1}} g(x)$$

$$= \sum_{x=2^n+1\ odd}^{2^{n+1}} 1 + \sum_{x=2^n+1\ odd}^{2^{n+1}} g(x+1) + \sum_{x=2^n+1\ even}^{2^{n+1}} g(x)$$

$$= 2^{n-1} + \sum_{x=2^n+1\ odd}^{2^{n+1}} g(x+1) + \sum_{x=2^n+1\ even}^{2^{n+1}} g(x)$$

$$= 2^{n-1} + \sum_{x=2^n+1\ even}^{2^{n+1}} g(x) + \sum_{x=2^n+1\ even}^{2^{n+1}} g(x)$$

$$= 2^{n-1} + 2\left(\sum_{x=2^n+1\ even}^{2^{n+1}} g(x)\right)$$

$$= 2^{n-1} + 2\left(\sum_{x=2^n+1\ even}^{2^{n+1}} (1 + g(x/2))\right)$$

$$= 2^{n-1} + 2\left(\sum_{x=2^n+1\ even}^{2^{n+1}} 1\right) + 2\left(\sum_{x=2^n+1\ even}^{2^{n+1}} g(x/2)\right)$$

$$= 2^{n-1} + 2\left(2^{n-1}\right) + 2\left(\sum_{x=2^n+1\ even}^{2^{n+1}} g(x/2)\right)$$

$$= 2^{n-1} + 2\left(2^{n-1}\right) + 2\left(\sum_{x=2^{n-1}+1}^{2^n} g(x)\right)$$

$$= 2^{n-1} + 2\left(2^{n-1}\right) + 2h(n-1)$$

$$= 3\left(2^{n-1}\right) + 2h(n-1).$$

From the recurrence $h(n) = 3\left(2^{n-1}\right) + 2h(n-1)$, we will try to obtain a closed form. By inspection, a formula of the form $(a + bn)2^{n-1}$ seems suitable for $h(n)$; plugging this in gives the following:

$h(n) = 3\left(2^{n-1}\right) + 2h(n-1)$
$\implies (a + bn)2^{n-1} = 3\left(2^{n-1}\right) + 2\left((a + b(n-1))2^{n-2}\right)$
$\implies (a + bn)2^{n-1} = 3\left(2^{n-1}\right) + (a + b(n-1))2^{n-1}$
$\implies a + bn = 3 + a + b(n-1)$
$\implies a + bn = 3 + a + bn - b$
$\implies 0 = 3 - b$
$\implies b = 3$

We also have the base case $h(0) = 1$ by simply evaluating from the definition; plugging this in gives us $h(0) = (a + 3(0))2^{0-1} = 1$, giving $a = 2$. It can be easily verified that $h(n) = (2 + 3n)2^{n-1}$ satisfies the base case and recurrence, so this closed form holds for all integer $n \geq 0$.

We want to compute $\sum_{x=2}^{2^{2025}} g(x) = h(0) + h(1) + h(2) + h(3) + \cdots + h(2024)$. Using our closed form, this evaluates to the sum $\sum_{n=0}^{2024} (2 + 3n)2^{n-1}$.

Note: We have $\sum_{n=0}^{2024} n2^{n-1} = \sum_{n=1}^{2024} 2^{n-1} + \sum_{n=2}^{2024} 2^{n-1} + \cdots + \sum_{n=2024}^{2024} 2^{n-1}$

$$
\begin{aligned}
&= \left(2^{2024} - 2^0\right) + \left(2^{2024} - 2^1\right) + \cdots + \left(2^{2024} - 2^{2023}\right)\\
&= 2024\left(2^{2024}\right) - \left(2^0 + 2^1 + \cdots + 2^{2023}\right)\\
&= 2024\left(2^{2024}\right) - \left(2^{2024} - 1\right)\\
&= 2023\left(2^{2024}\right) + 1.
\end{aligned}
$$

Thus our desired sum $\sum_{n=0}^{2024} (2+3n)2^{n-1}$ is computed as follows:

$$\sum_{n=0}^{2024} (2+3n)2^{n-1} = \sum_{n=0}^{2024} 2^n + 3\sum_{n=0}^{2024} n2^{n-1} = \left(2^{2025} - 1\right) + 3\left(2023\left(2^{2024}\right) + 1\right).$$

Now we just need to take this quantity mod $999 = 27 \times 37$. First, note that $2^{\phi(37)} = 2^{36} \equiv 1 \pmod{37}$, so:

$$2^{2024} = 2^8 \times 2^{2016} = 2^8 \times \left(2^{36}\right)^{56} \equiv 2^8 \times 1^{56} \equiv 2^8 \pmod{37}.$$

We also have that $2^{\phi(27)} = 2^{18} \equiv 1 \pmod{27}$, so:

$$2^{2024} = 2^8 \times 2^{2016} = 2^8 \times \left(2^{18}\right)^{112} \equiv 2^8 \pmod{27}.$$

Thus $2^{2024} \equiv 2^8 \pmod{999}$. It is straightforward to compute our desired quantity now:

$$
\begin{aligned}
&\left(2^{2025} - 1\right) + 3\left(2023\left(2^{2024}\right) + 1\right)\\
&= \left(2\left(2^{2024}\right) - 1\right) + 3\left(2023\left(2^{2024}\right) + 1\right)\\
&\equiv \left(2\left(2^8\right) - 1\right) + 3\left((25)\left(2^8\right) + 1\right) \pmod{999}\\
&\equiv 511 + 3(6401) \pmod{999}\\
&\equiv 511 + 19203 \pmod{999}\\
&\equiv 19714 \pmod{999}\\
&\equiv 733 \pmod{999}.
\end{aligned}
$$

Based on this computation, our answer is **733**.