UVAMT 2025 - Individual Round

I1. A two-digit positive integer x has the property that both $\sqrt{x}$ and $\sqrt[3]{x}$ are integers. Find x.
Solution: We need x to be a perfect square (since $\sqrt{x}$ is an integer) and a perfect cube (since $\sqrt[3]{x}$ is an integer). The two-digit perfect squares are 16, 25, 36, 49, 64, and 81; the two-digit perfect cubes are 27 and 64. The number 64 is the only one that appears on both lists, so the answer is **64**.

I2. A rectangle has perimeter 70 and area 300. How long is its diagonal?

Solution: Suppose the rectangle has width *w* and height *h*. We want the length of the diagonal *d*: by the Pythagorean Theorem, we have $d^2 = w^2 + h^2$, so $d = \sqrt{w^2 + h^2}$.

Since the perimeter is 70, we know that:
$2w + 2h = 70$
$\implies w + h = 35$
$\implies (w + h)^2 = 35^2$
$\implies w^2 + 2wh + h^2 = 1225$

Since the area is 300, we also know that:
$wh = 300$
$\implies w^2 + 2(300) + h^2 = 1225$
$\implies w^2 + h^2 = 1225 - 2(300)$
$\implies w^2 + h^2 = 625$
$\implies d = \sqrt{w^2 + h^2} = \sqrt{625} = 25.$

Therefore, the length of the diagonal *d* is **25.**

I3. Utkarsh comes across some snowballs in a field. Every minute, Utkarsh choose three snowballs and combines them into one. Suppose that after 20 minutes, there are 25 separate snowballs left in the field. How many snowballs must there have been at the start?

Solution: After every minute, 3 snowballs get replaced by 1, so the number of snowballs goes down by 2 every minute. So after 20 minutes, the number of snowballs has decreased by $2(20) = 40$. Since 25 snowballs remain at the end, that means the initial number of snowballs must have been $25 + 40 =$ **65**.

I4. Utkarsh and Vincent are running laps around a long circular track with circumference 21 miles. Utkarsh runs at a constant pace of 6 minutes per mile, while Vincent runs at a constant pace of 8 minutes per mile in the opposite direction. If they started at the same point, how many miles did Vincent run before passing Utkarsh for the first time?

Solution: Suppose that it took m minutes for Vincent to pass Utkarsh for the first time.

In those m minutes, Vincent ran $m/8$ miles (since he takes 8 minutes to run 1 mile), and Utkarsh ran $m/6$ miles (since he takes 6 minutes to run 1 mile). Also, at the instant that they first meet again, their combined distance traveled equals the circumference of the track, 21 miles. Then:

$$\frac{m}{8} + \frac{m}{6} = 21$$

$$\implies \frac{3m}{24} + \frac{4m}{24} = 21$$

$$\implies \frac{7m}{24} = 21$$

$$\implies 7m = 21(24) = 504$$

$$\implies m = \frac{504}{7} = 72$$

The total distance that Vincent ran is therefore $m/8 = 72/8 =$ **9** miles.

I5. Suppose there are 31 people signed up for UVAMT. Teams have a maximum size of 6, but any set of teams containing a total of at most 6 people can be combined together. However, teams cannot be broken up. Find the maximum possible number of teams (totaling 31 people) such that it is not possible to combine any of them.

Solution: Eight teams are possible, with team sizes of 4, 4, 4, 4, 4, 4, 4, 3. To show that nine teams aren't possible, notice that most one team can have a size of 3 or less (otherwise you could combine them). So if nine teams were possible, at least eight of them would need to have 4 or more members. This is a total of $4(8) = 32$ people, which is too many - this means nine is impossible, so the maximum possible amount of teams is **8**.

I6. Vincent has 6 buckets, with capacities 2, 3, 4, 5, 6, and 7 pounds. He also has 6 bricks, weighing 1, 2, 3, 4, 5, and 6 pounds. He wants to put exactly one brick in each bucket such that no bucket is above capacity. How many ways are there for him to accomplish this?

Solution: First, place the 6-pound brick: there are two possible buckets to put it in (the buckets with capacity 6 and 7 pounds). Next, place the 5-pound brick: there are still two possible buckets (the buckets with capacities 5, 6, and 7 pounds all work, but one of them is already occupied by the previous brick). Similarly, there are two possible buckets for the 4-, 3-, and 2-pound bricks, if you place them in that order. Finally, the 1-pound brick only has one remaining bucket to go in. The total number of possibilities is $2\times2\times2\times2\times2\times1 = \mathbf{32}$.

I7. In Chickenville, each household has either 1, 2, or 3 children. 20% of the children are only children, but 35% of households have only one child. Determine the average number of children per household.

Solution: Setting up a system of equations will work. But there's a better way with dimensional analysis:

$$\text{Average Children Per Household} = \frac{\text{Total Children}}{\text{Total Households}}$$

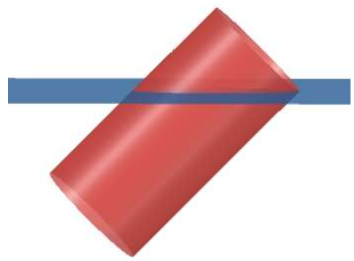
$$= \frac{\text{Number Of One-Child Households}}{\text{Total Households}} \times \frac{\text{Number Of Only Children}}{\text{Number Of One-Child Households}} \times \frac{\text{Total Children}}{\text{Number Of Only Children}}$$

$$= \frac{\text{Number Of One-Child Households}}{\text{Total Households}} \times \frac{\text{Number Of Only Children}}{\text{Number Of One-Child Households}} \div \frac{\text{Number Of Only Children}}{\text{Total Children}}$$

$$= 0.35 \times 1 \div 0.2$$

$$= \mathbf{1.75.}$$

I8. A cylindrical container with radius 1 and height 4 is partially filled with water. If tilted 45 degrees from the upright position, no water will spill, but if tilted any more than that, water will start to spill. Determine the volume of water in the cup.



The key to this problem is having a good diagram, as shown above. The bottom half of the cylinder is completely below the water level, contributing a volume of $\pi r^2 h = \pi(1)^2(2) = 2\pi$. If we only look at the top half of the cylinder, exactly half of this portion is above the water level, since the part above the water level forms the same 3D shape as the part below. So this part contributes half its volume: $\frac{\pi r^2 h}{2} = \frac{\pi(1)^2(2)}{2} = \frac{2\pi}{2} = \pi$. So the total volume is $2\pi + \pi = \mathbf{3\pi}$.

I9. A random number x is selected uniformly in $[0, 1]$. Mikhail can then apply as many moves to x as he wants, where a move is defined as rounding x to the decimal position of his choice. Mikhail wins if he can set x to 1 in finitely many moves. Find Mikhail's probability of winning.

Solution: Mikhail can win if and only if x is greater than 4/9. If it is greater than 4/9, then the first decimal digit of x excluding 4s is at least 5. Using this fact, the optimal strategy for Mikhail straightforward, as shown below:

$0.44462 \rightarrow 0.445 \rightarrow 0.45 \rightarrow 0.5 \rightarrow 1$

If x is less than or equal to 4/9, the first decimal digit of x excluding 4s is at most 3. Because of this, however Mikhail proceeds, he will eventually get stuck, as shown below:

$0.444348 \rightarrow 0.44435 \rightarrow 0.4444 \rightarrow$ All subsequent moves will only decrease the number!
$0.4419 \rightarrow 0.442 \rightarrow$ All subsequent moves will only decrease the number!

Therefore, he wins whenever x is between 4/9 and 1, which happens with probability **5/9**.

I10. A rectangular prism with positive integer side lengths has volume V and surface area $2V$. What is the sum of all possible values of V?

Solution: If the rectangle has positive integer side lengths a, b, c, then we need to solve the following equations:

Volume = $V = abc$
Surface Area = $2V = 2(ab + ac + bc) \implies V = ab + ac + bc$

Setting these equal to each other, we have that $abc = ab + ac + bc$.

Now, we'll divide both sides by abc to get $1 = 1/a + 1/b + 1/c$.

Clearly, a, b, and c all have to be greater than 1 - otherwise, the right side would be too big.

If a, b, and c are all at least 3, then the only solution is $a = 3$, $b = 3$, $c = 3$. Setting any of the variables higher would make the right side too small.

Otherwise, one of a, b, and c is less than 3. Since we know they all have to be greater than 1, one of them has to equal 2. Assume without loss of generality that a = 2 and $b \leq c$.

- If $b = 2$, then the right side becomes too big.
- If $b = 3$, then $c = 6$.
- If $b = 4$, then $c = 4$.
- If b $\geq$ 5, then since we assumed $b \leq c$, the right side would become too small.

Therefore, the only solutions up to a permutation of the side lengths are (a, b, c) = (3, 3, 3), (2, 3, 6), and (2, 4, 4). Therefore, the possible values of $V = abc$ are:

- If (a, b, c) = (3, 3, 3), then $V = 3\times3\times3 = 27$.
- If (a, b, c) = (2, 3, 6), then $V = 2\times3\times6 = 36$.
- If (a, b, c) = (2, 4, 4), then $V = 2\times4\times4 = 32$.

Therefore, the sum of all possible values of V is 27+36+32 = **95**.

I11. Suppose that sets $S_1, S_2, \cdots, S_{12}$ satisfy the following properties:

- Any three of these sets have exactly one element in common.
- Any four of these sets have no elements in common.

Find the minimum possible value of $|S_1|$.

Solution: For each possible combination of three sets, we need to add a unique element into those sets. The condition that any four of these sets have no elements in common makes it so that this element must be unique - it can't be reused for any other combination of three sets. Using this construction, the number of elements in S_1 is just the number of ways to choose three sets among $S_1, S_2, \cdots, S_{12}$ such that S_1 is chosen. Since S_1 must be chosen, we have to choose two other sets among the remaining sets $S_2, \cdots, S_{12}$, so there are $\binom{11}{2} = 55$ elements in S_1. All of these elements were forced to be added (there is no way we could have avoided adding them), so the minimum possible size of S_1 is **55**.

Note: Larger sets satisfying the constraints are possible, by adding single elements to S_1 not present in any other set. However, 55 is still minimal.

I12. Let S be a set of integers. Suppose that for any two distinct integers $x, y \in S$ and any nonnegative integer k, we have $2^k x \not\equiv y \pmod{2^{12} - 1}$. Find the maximum possible size of S.

Solution: Without loss of generality, assume that S only contains numbers between 0 (inclusive) and $2^{12} - 1$ (exclusive). Express x as a number in binary of length 12, left-padding with zeros as needed. Then the operation of multiplying it by 2 and reducing mod $2^{12} - 1$ is equivalent to cycling all the bits left by one position: each bit moves left by one position; if a 1 bit moves into the 2^{12}-position, then that 1 goes to the end of the number (because $2^{12} \equiv 2^0 \bmod 2^{12} - 1$). Then multiplying x by 2^k is equivalent to doing this operation k times; that is, cycling all the bits left by k positions. Therefore, if two numbers are cyclic shifts of each other when written as length-12 binary numbers (that is, you can reach one number from the other by cycling the bits by some number of positions), they cannot both be included. On the other hand, if two numbers x, y between 0 (inclusive) and $2^{12} - 1$ (exclusive) are not cyclic shifts of each other, there cannot exist an integer k such that $2^k x \equiv y \pmod{2^{12} - 1}$. Therefore, the answer is just the number of length-12 binary strings with cyclic shifts considered equivalent, minus 1 because the all-ones binary string is not in the range 0 (inclusive) to $2^{12} - 1$ (exclusive).

This quantity can be computed using Burnside's Lemma.

For each rotation k (from 0 to 11), we count how many 12-bit strings stay the same when their bits are rotated left by k positions. Burnside's Lemma tells us that the average of these values equals the quantity we're looking for.

The number of such fixed strings depends on the greatest common divisor of 12 and k. Specifically, the number of 12-bit strings fixed under rotation by k positions is $2^{\gcd(12,k)}$. This is because the string must repeat every $\gcd(12, k)$ positions to be unchanged by rotation.

Let's compute this sum:

- $k = 0: \ 2^{\gcd(12,k)} = 2^{12} = 4096$

- $k = 1: 2^{\gcd(12,k)} = 2^1 = 2$
- $k = 2: 2^{\gcd(12,k)} = 2^2 = 4$
- $k = 3: 2^{\gcd(12,k)} = 2^3 = 8$
- $k = 4: 2^{\gcd(12,k)} = 2^4 = 16$
- $k = 5: 2^{\gcd(12,k)} = 2^1 = 2$
- $k = 6: 2^{\gcd(12,k)} = 2^6 = 64$
- $k = 7: 2^{\gcd(12,k)} = 2^1 = 2$
- $k = 8: 2^{\gcd(12,k)} = 2^4 = 16$
- $k = 9: 2^{\gcd(12,k)} = 2^3 = 8$
- $k = 10: 2^{\gcd(12,k)} = 2^2 = 4$
- $k = 11: 2^{\gcd(12,k)} = 2^1 = 2$

Now add these up and divide by 12: (4096+2+4+8+16+2+64+2+16+8+4+2)/12 = 352.

Finally, we subtract 1, because we must exclude the class containing the all-ones string (111111111111), since it corresponds to the number $2^{12} - 1 = 4095$, which is out of our range. Therefore, the maximum possible size of S is **351**.