# UVAMT 2025 - Team Round

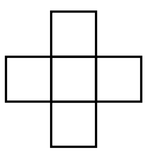
T1. On a table, some coins are placed in a row, each showing heads or tails, such that:

- Exactly 3 coins are showing tails,
- Each tail is adjacent to two heads, and
- Each head is adjacent to exactly one other head.

How many coins must there be on the table in total?

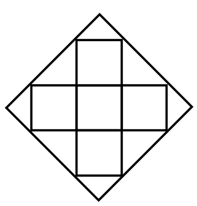
Solution: In a valid arrangement, each of the tails must be sandwiched between pairs of adjacent heads. This gives us the arrangement H H T H H T H H T H H, which contains a total of **11** coins.

T2. Remove four 1x1 squares from the corners of a 3x3 square to obtain the following figure:

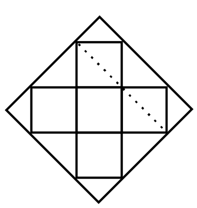


It turns out that this figure fits in a square smaller than 3x3! What's the minimum side length of a square that this figure can fit in?

Solution: To achieve this smaller fit, use a square rotated by 45 degrees:



In the figure below, notice that the dotted line has a length of twice the diagonal of one of the smaller squares, and has the same length as a side of the large square:



The length of the diagonal in a square of side length 1 is $\sqrt{1^2+1^2}=\sqrt{2}$, by the Pythagorean theorem. Therefore, the side length of the large square is $\mathbf{2\sqrt{2}}$.

T3. Mikhail wants to fill in the empty cells in this grid with integers, such that every row, column, and 3-cell diagonal sums to the same value $v$. Find $v$.

| | | |
|---|---|---|
| | | 4 |
| 3 | | |
| | 5 | 8 |

Solution: We'll assign variables to the unknown cells for extra clarity:

| | | |
|---|---|---|
| *(a)* | *(b)* | 4 |
| 3 | *(c)* | *(d)* |
| *(e)* | 5 | 8 |

From the middle row and rightmost column, we have $3+c+d = 4+d+8$, so cancelling $d$ from both sides we get $3+c = 4+8$. Therefore, $c = 9$.

From the bottom row and leftmost column, we have $e+5+8 = a+3+e$, so cancelling $e$ from both sides we get $5+8 = a+3$. Therefore, $a = 10$.

Finally, from the diagonal containing $a$, $c$, and 8, we get $v = a+c+8$, which equals 10+9+8=**27**.

Note: from here, it's relatively straightforward to fill in the grid, even if not strictly necessary to solve the problem:

| | | |
|---|---|---|
| 10 | 13 | 4 |
| 3 | 9 | 15 |
| 14 | 5 | 8 |

As we wanted, all rows, columns, and diagonals in the completed grid add up to **27**.

T4. How many 4-digit numbers have exactly 3 odd digits?

Solution: We will split into cases, indicating which position contains the one even digit.

- If the first digit is even, then there are 4 choices for it (2, 4, 6, or 8), and 5 choices for

each of the other three odd digits (1, 3, 5, 7, or 9). Note that the first digit cannot be zero. This gives a total of $4 \times 5 \times 5 \times 5 = 500$ solutions.
- If the second digit is even, then there are 5 choices for it (0, 2, 4, 6, or 8), and 5 choices for each of the other three odd digits (1, 3, 5, 7, or 9). (The difference between this case and the previous one is that non-leading digits are allowed to be zero.) This gives a total of $5 \times 5 \times 5 \times 5 = 625$ solutions.
- If the third digit is even, we get another 625 solutions, using the same logic as the previous case.
- Finally, if the last digit is the even one, we get another 625 solutions.

The total number of solutions is therefore $500+625+625+625 =$ **2375**.

T5. Suppose that $x$ is an integer such that both $x$ and $x + 45$ are perfect squares. Determine the sum of all possible values of $x$.

Solution: We will set $x = a^2$ and $x + 45 = b^2$. Then:

$$b^2 - a^2 = (x + 45) - x = 45$$
$$\implies (b + a)(b - a) = 45$$

The possible integer factor pairs $(b + a, b - a)$ of 45 are (45, 1), (15, 3), and (9, 5).

- If $(b + a, b - a) = (45, 1)$, then:
  $(b + a) - (b - a) = 44 \implies 2a = 44 \implies a = 22 \implies a^2 = 484 \implies x = 484.$
- If $(b + a, b - a) = (15, 3)$, then:
  $(b + a) - (b - a) = 12 \implies 2a = 12 \implies a = 6 \implies a^2 = 36 \implies x = 36.$
- If $(b + a, b - a) = (9, 5)$, then:
  $(b + a) - (b - a) = 4 \implies 2a = 4 \implies a = 2 \implies a^2 = 4 \implies x = 4.$

The sum of all possible values of $x$ is $484 + 36 + 4 =$ **524**.

(Technical note: you can get more possible factor pairs if you swap d and e, or negate them both, but if you follow the same process with these solutions, you won't get any new possible values of $x$.)

T6. Thirty students are in a school that offers three courses. Each course is taken by exactly 21 students. Find the maximum possible number of students taking exactly one course.

Solution: Assign the following variables.

- $x$ = number of students taking exactly one course
- $y$ = number of students taking exactly two courses
- $z$ = number of students taking exactly three courses.

Since there are 30 students, we know that $x + y + z \leq 30$. Rearrange this to get $y \leq 30 - x - z$.

Each course has 21 students, so total enrollment is $21 \times 3 = 63$. So we can also deduce that
$x + 2y + 3z = 63$.

Substituting, we get:

$$\begin{aligned} 63 &= x + 2y + 3z \\ &\leq x + 2(30 - x - z) + 3z \\ &= 60 - x + z. \end{aligned}$$

Therefore:

$$\begin{aligned} &63 \leq 60 - x + z \\ &\implies 3 \leq -x + z \\ &\implies 3 + x \leq z \\ &\implies z \geq 3 + x. \end{aligned}$$

If $x$ were at least 14, then $z$ would have to be at least 17, giving a total of 31 students, which is too many. Therefore, $x$ has to be at most 13.

To see that this is possible to achieve, we use the following allocation:

- 4 students taking only course A
- 4 students taking only course B
- 5 students taking only course C
- 1 student taking courses A and B
- 16 students taking courses A, B, and C

This arrangement totals 30 students, and exactly 21 students are taking each course. A total of 13 students are taking only one course, so the maximum value of 13 is achievable. Since it is impossible to do better, the maximum possible number is **13**.

T7. Let a positive integer $n$ be *satisfying* if there exists a geometric progression $a, b, c$ of positive integers $a < b < c$ with $a + b + c = n$. Find the smallest $n$ such that both $n$ and $n + 2$ are satisfying.

Solution: We'll take cases on the common ratio *r*. Note that since $a < b < c$, this common ratio must be (strictly) greater than 1.

- r = 2: the geometric progression is of the form x, 2x, 4x, for some positive integer x. Their sum is 7x, so the numbers 7, 14, 21, 28, ... are all satisfying.
- r = 3: the geometric progression is of the form x, 3x, 9x, for some positive integer x. Their sum is 13x, so the numbers 13, 26, 39, ... are all satisfying.
- r = 4: the geometric progression is of the form x, 4x, 16x, for some positive integer x. Their sum is 21x, so the numbers 21, 42, 63, ... are all satisfying.
- r = 3/2: the geometric progression is of the form 4x, 6x, 9x, for some positive integer x. Their sum is 19x, so the numbers 19, 38, 57 ... are all satisfying.

From these lists, we see that 19 and 21 are both satisfying, so $n = 19$ satisfies the conditions of the problem (both $n$ and $n+2$ are satisfying). It's not hard to see that any larger or more complex value of $r$ won't give you any satisfying numbers smaller than 19. So, the value of $n$ we found is optimal, so the answer is **19**.

T8. Find the sum of all real numbers $a < 2025$ such that $x\lfloor x \rfloor = a$ has at least two solutions.

Solution: First, notice that the function $x\lfloor x \rfloor$ is increasing for $x > 0$, and decreasing for $x < 0$.

We consider the range of $x\lfloor x \rfloor$ when $x > 0$. When $x$ is an integer, $x\lfloor x \rfloor$ is a perfect square. But if $x$ decreases by an arbitrarily small amount to $x'$, then $x'\lfloor x' \rfloor$ becomes arbitrarily close to $x(x-1)$. So, there's a discontinuous jump in the values that the function attained from $x(x-1)$ to $x^2$, for all positive integer $x$. Note that the value $x(x-1)$ is never actually attained, though the value $x^2$ is. Therefore, using this fact about where the discontinuous jumps are, and noting that the function is continuous everywhere else, the range of $x\lfloor x \rfloor$ when $x > 0$ is $0 \cup [1,2) \cup [4,6) \cup [9,12) \cup [16,20) \cdots$.

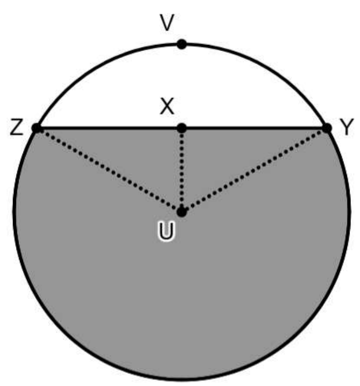
Next, we consider the range of $x\lfloor x \rfloor$ when $x < 0$. By the same logic as before, there's a discontinuous jump in the values that the function attained from $x(x-1)$ to $x^2$. But now, the values attained are different: using this fact about where the discontinuous jumps are, the range of $x\lfloor x \rfloor$ when $x < 0$ is $\cdots [25,20) \cup [16,12) \cup [9,6) \cup [4,2) \cup [1,0)$.

For any nonzero value of $a$, the equation $x\lfloor x \rfloor = a$ has at most one solution for which $x > 0$, and at most one solution for which $x < 0$ (this is because the function $x\lfloor x \rfloor$ is *strictly* increasing/decreasing over these intervals when $a$ is nonzero), so this equation has two solutions if and only if $a$ is in both of the ranges listed above - the range $x\lfloor x \rfloor$ when $x > 0$ and when $x < 0$. These values of $a$ are just the perfect squares. We can safely ignore the $a = 0$ case because it won't affect the sum. Therefore, the sum we want is just all the perfect squares below 2025:

namely, $1^2 + 2^2 + 3^2 + \cdots + 44^2$. This is equal to $\dfrac{44(44+1)(2(44)+1)}{6} = \mathbf{29370}$.

T9. Utkarsh and Vincent are moving to UVAMT grounds, which are shaped like a perfect circle. Utkarsh is at the center of grounds, and Vincent is at the edge. They pick a random spot uniformly within grounds to meet up. What is the probability that Utkarsh is closer to this spot than Vincent?

Solution: We will use the following diagram, where Utkarsh is located at point U, and Vincent at point V.




The set of points equidistant from U and V is the perpendicular bisector of segment UV. As such, we define line ZY to be the perpendicular bisector of segment UV.

Without loss of generality, let the radius of this circle be 1 unit. Then the shaded area divides into a circular sector and a triangle ZUY.

First, we will find the area of triangle ZUY. Its height UX is 1/2, since UV = 1 and X is the midpoint of U and V (because ZY was defined as the perpendicular bisector). To find the base, notice that UY = 1 and UX = 1/2, so triangle XUY is a 30-60-90 triangle, so XY = $\sqrt{3}/2$. Since X is the midpoint of ZY, we have ZY = 2XY = $\sqrt{3}$. Therefore, the area of triangle ZUY is $(1/2)\left(\sqrt{3}\right)(1/2) = \sqrt{3}/4$.

Now we will find the area of the circular sector. From before, we had that triangle XUY is a 30-60-90 triangle. So $\angle$ZUX = $\angle$XUY = 60°, so the circular sector occupies the remaining 360° - 60° - 60° = 240°, or 2/3 of the circle. Thus the area of this circular sector is $(2/3)\pi$.

Since the random spot is picked uniformly randomly inside the circle, the probability we want is just the area of the shaded region (denoting all points closer to U than to V) divided by the area of the circle ($\pi$). This is equal to $\frac{(2/3)\pi + \sqrt{3}/4}{\pi} = \mathbf{\frac{2}{3} + \frac{\sqrt{3}}{4\pi}}$.

T10. How many non-constant polynomials with integer coefficients pass through the point $(-2, 27)$ and have all positive integer roots?

Solution: Let $p(x) = a(x - r_1)(x - r_2) \cdots (x - r_d)$ be a degree *d* polynomial that satisfies the constraints of the problem. We need $p(-2) = a(-2 - r_1)(-2 - r_2) \cdots (-2 - r_d) = 27$. Therefore, the product $(-2 - r_1)(-2 - r_2) \cdots (-2 - r_d)$ must be a factor of 27, up to sign. Notice that each of the linear factors $(-2 - r_1), (-2 - r_2), \ldots$ must be -3 or less, which will greatly restrict how many factors we can have. We will take cases on the degree *d*.

Case 1 (d=1): We have one linear factor that could evaluate to -3, -9, or -27. This gives us the following polynomials:

- $p(x) = -9(x - 1)$: linear factor evaluates to -3
- $p(x) = -3(x - 7)$: linear factor evaluates to -9
- $p(x) = -(x - 25)$: linear factor evaluates to -27

Case 2 (d=2): We need two linear factors; they could evaluate to (-3)(-3) or (-3)(-9). This gives us the following polynomials:

- $p(x) = 3(x - 1)(x - 1)$: linear factors evaluate to (-3)(-3) = 9
- $p(x) = (x - 1)(x - 7)$: linear factors evaluate to (-3)(-9) = 27

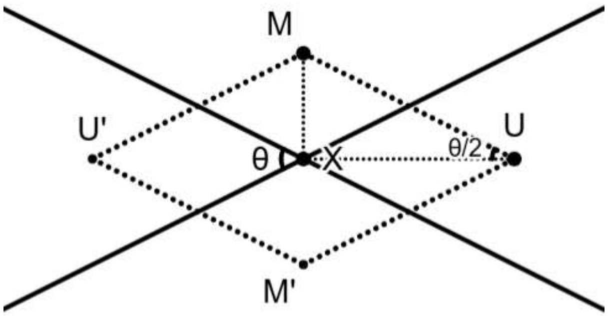
Case 3 (d=3): We need three linear factors; they must all evaluate to -3 (otherwise the product would exceed 27 in absolute value). This gives us the following polynomials:

- $p(x) = -(x - 1)(x - 1)(x - 1)$: linear factors evaluate to (-3)(-3)(-3) = -27

If the degree exceeds 3, then the product would get too large (its absolute value would be at least $3^4 = 81$, which is greater than 27. So, this covers all cases, giving a total of **6** polynomials.

T11. Two straight roads intersect at an angle $\theta$. Utkarsh and Mikhail are both $1$ mile away from each of the roads, but Utkarsh is twice as far from their intersection as Mikhail is. Find $\cos\theta$. Note: You may assume that the roads are infinitely long in both directions.

Solution: Consider the following diagram, where U represents Utkarsh's position, M represents Mikhail's position, and the two intersecting roads are denoted as the solid lines.




We reflect U across X to obtain the point U', and reflect M across X to obtain the point M'. Since Each of M, U, M', U' are one mile away from the roads, so each side MU, UM', M'U', U'M measures exactly two miles: two of these one mile distances joined together. Thus MUM'U' is a rhombus. Diagonals of a rhombus are perpendicular, so ∠MXU is a right angle.

Since the dotted lines are parallel with their corresponding solid lines (since the two points they're defined by are equidistant from the road), we have that ∠MUM' = $\theta$. Thus ∠MUX = $\theta/2$.

Since Utkarsh is twice as far from their intersection as Mikhail is, we have UX = 2MX. Thus UM = $\sqrt{5}$MX by the Pythagorean Theorem. Thus
$\cos(\theta/2) = \text{UX}/\text{UM} = 2\text{MX}/\sqrt{5}\text{MX} = 2/\sqrt{5}$, so
$\cos\theta = 2\cos^2(\theta/2) - 1 = 2\left(2/\sqrt{5}\right)^2 - 1 = 2(4/5) - 1 = \mathbf{3/5}$.

Note: $\theta$ could also be taken as the other possible angle of intersection between the two roads; $\mathbf{-3/5}$ was also an acceptable answer.

T12. Eight vertices are placed in 3D space to form a unit cube. Utkarsh chooses four distinct vertices randomly out of these eight. What is the expected volume of the tetrahedron formed by these four vertices? (Note: if all four vertices lie on the same plane, the volume is said to be 0 by convention.)

Solution: There are $\binom{8}{4} = 70$ distinct ways to choose four vertices out of 8, and each of these choices are equally likely. We will take cases on the type of tetrahedron formed:

Case 1: All four vertices lie on the same plane. Then either they lie on the same face (6 cases), or they are formed by taking the vertices among two pairs of opposite edges (6 cases, because the 12 edges of the cube form 6 pairs). This gives 12 cases total, each with a volume of 0.

Case 2: The four vertices form a regular tetrahedron. This happens when you choose alternating vertices - that is, a set of vertices where no two of them share an edge. The only such possibilities are $\{(0,0,0),(0,1,1),(1,1,0),(1,0,1)\}$ and $\{(0,0,1),(0,1,0),(1,0,0),(1,1,1)\}$. This gives 2 cases total. The volume of a regular tetrahedron with side length x is $x^3 / \left(6\sqrt{2}\right)$, and the side length of these tetrahedra are $\sqrt{2}$, giving a volume of $\frac{\sqrt{2}^3}{6\sqrt{2}} = \frac{2\sqrt{2}}{6\sqrt{2}} = \frac{1}{3}$. (Alternatively, these regular tetrahedra can be thought of as taking a cube and removing four right-angled tetrahedra from the corners not selected; each of these right-angled tetrahedra has base area 1/2 and height 1, giving a volume of (1/2)(1)/3 = 1/6, so the remaining volume is 1 - 4(1/6) = 1/3.)

Case 3: The remaining 70 - 12 - 2 = 56 cases all contain one face with three selected vertices (the last selected vertex is on the opposite face). If we consider the face with three selected vertices to form the base of our tetrahedron, the base has area 1/2, and the height of the tetrahedron is 1, so its volume is (1/2)(1)/3 = 1/6.

Therefore, the expected volume is (12/70)(0) + (2/70)(1/3) + (56/70)(1/6) = 1/7.

T13. Utkarsh selects an integer between 1 and $n$ inclusive, and Vincent makes a series of guesses to try to determine Utkarsh's number. When he makes a guess, Utkarsh tells him whether it was greater than, less than, or equal to the correct number. If it was greater, Utkarsh gets 3 points; if it was less, Utkarsh gets 1 point; if it was correct, the game ends. Find the greatest $n$ such that Vincent can ensure Utkarsh scores at most 10 points.

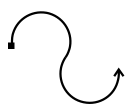
Solution: We use recursion. Let f(x) be the greatest *n* such that Vincent can ensure Utkarsh scores at most *x* points. It's clear from testing small cases that f(0) = 1, f(1) = 2, and f(2) = 3. For x at least 3, Vincent's optimal strategy is to make a guess such that if it was greater than the correct number, f(x-3) possible numbers remain (since he would have 3 fewer points to spare in this case), and if it was less than the correct number, f(x-1) possible numbers remain (since he would have 1 fewer point to spare in this case). Therefore, the function f(x) satisfies the recurrence f(x) = f(x-3) + f(x-1) + 1, where the +1 represents the case where Vincent guesses correctly. Calculating out this recurrence, we get:

- f(3) = f(0) + f(2) + 1 = 1 + 3 + 1 = 5
- f(4) = f(1) + f(3) + 1 = 2 + 5 + 1 = 8
- f(5) + f(2) + f(4) + 1 = 3 + 8 + 1 = 12
- f(6) = f(3) + f(5) + 1 = 5 + 12 + 1 = 18
- f(7) = f(4) + f(6) + 1 = 8 + 18 + 1 = 27

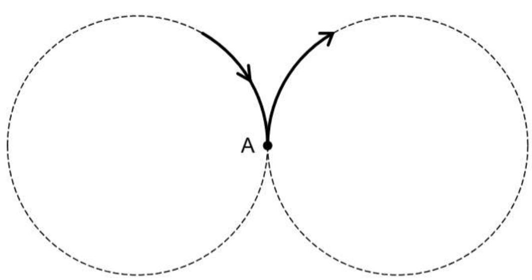
- f(8) = f(5) + f(7) + 1 = 12 + 27 + 1 = 40
- f(9) = f(6) + f(8) + 1 = 18 + 40 + 1 = 59
- f(10) = f(7) + f(9) + 1 = 27 + 59 + 1 = 87

Therefore, the greatest *n* such that Vincent can ensure Utkarsh scores at most 10 points is f(10) = **87**.

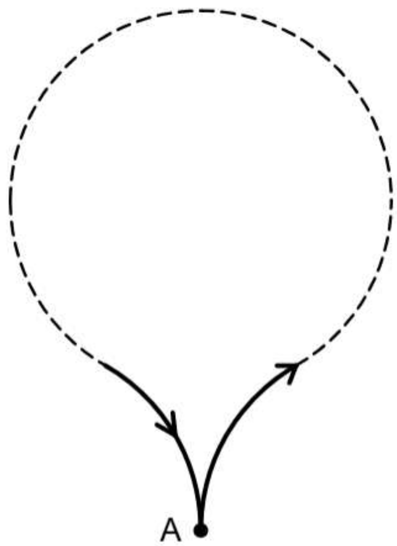
T14. You're driving a go-kart without gas or brakes - just a steering wheel. The go-kart travels in arcs of radius 1, and you can instantaneously change the direction of its arc between clockwise and counterclockwise, but this is all the control you have. You're currently at point $A$ traveling due north. What's the minimum distance you must travel to return to point $A$ traveling due south? (An example of the go-kart's movement pattern is given below.)



Solution: Although it's possible to go straight by switching between clockwise and counterclockwise arbitrarily quickly, it's not actually optimal to do that, because (on an intuitive level) the distance traveled while going straight could have been used to make progress. So, an optimal path should start and end as follows:




Then, to connect these two partial paths together, the most efficient way is to loop around the top, as follows:




The initial and final portions (denoted by solid lines) should be the same length, so that they can connect together. The only free parameter in this setup is how long the initial/final portions should be. Suppose that, during the initial portion, the go-kart has turned $\theta$ degrees clockwise. Then, in the final portion, the go-kart also turns $\theta$ degrees clockwise. The total net rotation needs to be 180 degrees counterclockwise, so the portion in the dotted lines should be a turn of $180 + 2\theta$ degrees counterclockwise.

We need the x-coordinates to match up: that is, we want to end at A, not at some point to the left or right of A. An angle of $\theta = 60°$ works: the initial and final portions both result in a net x-coordinate change of +0.5 units, and the 300-degree portion results in a net x-coordinate change of -1 unit.

Therefore, the total path length is $\pi/3$ (initial portion) + $5\pi/3$ (300-degree portion) + $\pi/3$ (final portion) = $\mathbf{7\pi/3}$.

T15. For positive integer $x$, let $f(x) = \begin{cases} x/2 & (x \text{ even}) \\ x+1 & (x \text{ odd}) \end{cases}$. Let $g(x)$ be the minimum $k$ such that $\underbrace{f(f(\dots f(x)))}_{k} = 1$. Compute $\sum_{x=2}^{2^{2025}} g(x)$ mod 999.

Solution: Let $h(n) = \sum_{x=2^n+1}^{2^{n+1}} g(x)$. We will derive a recurrence relation for h(n).

Note that $g(x) = 1 + g(x/2)$ if x is even, and $g(x) = 1 + g(x+1)$ if x is odd.

Split $h(n)$ into its odd and even terms, and simplify as follows:

$$h(n) = \sum_{x=2^n+1\ odd}^{2^{n+1}} g(x) + \sum_{x=2^n+1\ even}^{2^{n+1}} g(x)$$

$$= \sum_{x=2^n+1\ odd}^{2^{n+1}} (1 + g(x+1)) + \sum_{x=2^n+1\ even}^{2^{n+1}} g(x)$$

$$= \sum_{x=2^n+1\ odd}^{2^{n+1}} 1 + \sum_{x=2^n+1\ odd}^{2^{n+1}} g(x+1) + \sum_{x=2^n+1\ even}^{2^{n+1}} g(x)$$

$$= 2^{n-1} + \sum_{x=2^n+1\ odd}^{2^{n+1}} g(x+1) + \sum_{x=2^n+1\ even}^{2^{n+1}} g(x)$$

$$= 2^{n-1} + \sum_{x=2^n+1\ even}^{2^{n+1}} g(x) + \sum_{x=2^n+1\ even}^{2^{n+1}} g(x)$$

$$= 2^{n-1} + 2\left(\sum_{x=2^n+1\ even}^{2^{n+1}} g(x)\right)$$

$$= 2^{n-1} + 2\left(\sum_{x=2^n+1\ even}^{2^{n+1}} (1 + g(x/2))\right)$$

$$= 2^{n-1} + 2\left(\sum_{x=2^n+1\ even}^{2^{n+1}} 1\right) + 2\left(\sum_{x=2^n+1\ even}^{2^{n+1}} g(x/2)\right)$$

$$= 2^{n-1} + 2\left(2^{n-1}\right) + 2\left(\sum_{x=2^n+1 \text{ even}}^{2^{n+1}} g(x/2)\right)$$

$$= 2^{n-1} + 2\left(2^{n-1}\right) + 2\left(\sum_{x=2^{n-1}+1}^{2^n} g(x)\right)$$

$$= 2^{n-1} + 2\left(2^{n-1}\right) + 2h(n-1)$$

$$= 3\left(2^{n-1}\right) + 2h(n-1).$$

From the recurrence $h(n) = 3\left(2^{n-1}\right) + 2h(n-1)$, we will try to obtain a closed form. By inspection, a formula of the form $(a+bn)2^{n-1}$ seems suitable for $h(n)$; plugging this in gives the following:

$$\begin{aligned}
&h(n) = 3\left(2^{n-1}\right) + 2h(n-1)\\
&\implies (a+bn)2^{n-1} = 3\left(2^{n-1}\right) + 2\left((a+b(n-1))2^{n-2}\right)\\
&\implies (a+bn)2^{n-1} = 3\left(2^{n-1}\right) + (a+b(n-1))2^{n-1}\\
&\implies a+bn = 3+a+b(n-1)\\
&\implies a+bn = 3+a+bn-b\\
&\implies 0 = 3-b\\
&\implies b = 3
\end{aligned}$$

We also have the base case $h(0) = 1$ by simply evaluating from the definition; plugging this in gives us $h(0) = (a+3(0))2^{0-1} = 1$, giving $a = 2$. It can be easily verified that $h(n) = (2+3n)2^{n-1}$ satisfies the base case and recurrence, so this closed form holds for all integer $n \geq 0$.

We want to compute $\sum_{x=2}^{2^{2025}} g(x) = h(0) + h(1) + h(2) + h(3) + \cdots + h(2024)$. Using our closed form, this evaluates to the sum $\sum_{n=0}^{2024} (2+3n)2^{n-1}$.

Note: We have $\sum_{n=0}^{2024} n2^{n-1} = \sum_{n=1}^{2024} 2^{n-1} + \sum_{n=2}^{2024} 2^{n-1} + \cdots + \sum_{n=2024}^{2024} 2^{n-1}$

$$= \left(2^{2024} - 2^0\right) + \left(2^{2024} - 2^1\right) + \cdots + \left(2^{2024} - 2^{2023}\right)$$
$$= 2024\left(2^{2024}\right) - \left(2^0 + 2^1 + \cdots + 2^{2023}\right)$$
$$= 2024\left(2^{2024}\right) - \left(2^{2024} - 1\right)$$
$$= 2023\left(2^{2024}\right) + 1.$$

Thus our desired sum $\sum_{n=0}^{2024} (2+3n)2^{n-1}$ is computed as follows:

$$\sum_{n=0}^{2024} (2+3n)2^{n-1} = \sum_{n=0}^{2024} 2^n + 3\sum_{n=0}^{2024} n2^{n-1} = \left(2^{2025} - 1\right) + 3\left(2023\left(2^{2024}\right) + 1\right).$$

Now we just need to take this quantity mod $999 = 27 \times 37$. First, note that $2^{\phi(37)} = 2^{36} \equiv 1 \pmod{37}$, so:

$$2^{2024} = 2^8 \times 2^{2016} = 2^8 \times \left(2^{36}\right)^{56} \equiv 2^8 \times 1^{56} \equiv 2^8 \pmod{37}.$$

We also have that $2^{\phi(27)} = 2^{18} \equiv 1 \pmod{27}$, so:

$$2^{2024} = 2^8 \times 2^{2016} = 2^8 \times \left(2^{18}\right)^{112} \equiv 2^8 \pmod{27}.$$

Thus $2^{2024} \equiv 2^8 \pmod{999}$. It is straightforward to compute our desired quantity now:

$$\left(2^{2025} - 1\right) + 3\left(2023\left(2^{2024}\right) + 1\right)$$
$$= \left(2\left(2^{2024}\right) - 1\right) + 3\left(2023\left(2^{2024}\right) + 1\right)$$
$$\equiv \left(2\left(2^8\right) - 1\right) + 3\left((25)\left(2^8\right) + 1\right) \pmod{999}$$
$$\equiv 511 + 3(6401) \pmod{999}$$
$$\equiv 511 + 19203 \pmod{999}$$
$$\equiv 19714 \pmod{999}$$
$$\equiv 733 \pmod{999}.$$

Based on this computation, our answer is **733**.