UVAMT 2026 Individual Round Solutions

I1. Answer: 81

Let A be the number in the first box, and B be the number in the second box.

$$100 - \boxed{A} = \boxed{B}$$

Since A has two digits, we have $A \geq 10$, and since B has two digits, we have $B \geq 10 \implies 100 - A \geq 10 \implies A \leq 90$.

This means that A must be some number in $\{10, 11, \ldots, 90\}$. Each value of A uniquely determines $B = 100 - A$, and one can verify B is also a two-digit integer in each case.

Therefore, to count all ways, it suffices to count the number of possible values of A. We can obtain the set of values $\{10, 11, \ldots, 90\}$ by starting out with the 90 numbers $\{1, 2, \ldots, 90\}$, and taking away the 9 numbers $\{1, 2, \ldots, 9\}$, giving $90 - 9 = \boxed{81}$ values.

I2. Answer: 10%

Let the number of face-down cards be d. Since there are exactly 50% more face-up cards than face-down cards, the number of face-up cards is $1.5d$. Adding these together, there are $2.5d$ cards in total.

We want equal numbers of face-up and face-down cards at the end, i.e., each group should be half the total, or $1.25d$ cards. This means we need to go from d face-down cards initially to $1.25d$ face-down cards at the end. Subtracting, this means we must flip a total of $0.25d$ cards over.

Therefore, the fraction of total cards flipped is:

$$\frac{0.25d}{2.5d} = \frac{0.25}{2.5} = \frac{1}{10} = \boxed{10\%}.$$

I3. Answer: 12

Let a be the number of 10-cent coins and b the number of 3-cent coins. We need:

$$10a + 3b = 50, \quad a \geq 1, \quad b \geq 1.$$

If there are five or more 10-cent coins, then their total value is at least 50 cents, leaving no room for any 3-cent coins. Therefore, there are at most four 10-cent coins, so $a = 1, 2, 3,$ or 4.

If $a = 1$: We have $10(1) + 3b = 50 \implies 3b = 40 \implies b = \frac{40}{3}$, which is not an integer.

If $a = 2$: We have $10(2) + 3b = 50 \implies 3b = 30 \implies b = \frac{30}{3} = 10$. This gives a valid solution: namely, $a = 2$ and $b = 10$.

If $a = 3$: We have $10(3) + 3b = 50 \implies 3b = 20 \implies b = \frac{20}{3}$, which is not an integer.

If $a = 4$: We have $10(4) + 3b = 50 \implies 3b = 10 \implies b = \frac{10}{3}$, which is not an integer.

Therefore, the only solution is to have two 10-cent coins ($a = 2$) and ten 3-cent coins ($b = 10$), for a total of $\boxed{12}$ coins.

I4. Answer: 5

Let the sum of the n readings be S. If we add a reading of 3°, then the new sum becomes $S+3$, over a total of $n+1$ readings, for an average of $\frac{S+3}{n+1}$. If we instead add a reading of 15°, then the new sum becomes $S+15$, over a total of $n+1$ readings, for an average of $\frac{S+15}{n+1}$. To summarize:

Condition 1 (adding a reading of 3°):

$$-1 \leq \frac{S+3}{n+1} \leq 1;$$

Condition 2 (adding a reading of 15°):

$$-1 \leq \frac{S+15}{n+1} \leq 1.$$

Since both of these fractions must lie between -1 and 1 inclusive, they may differ by at most 2. That is:

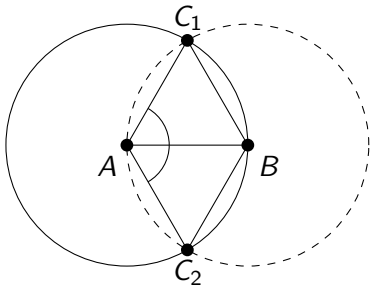
$$\frac{S+15}{n+1} - \frac{S+3}{n+1} \leq 2 \implies \frac{12}{n+1} \leq 2 \implies 12 \leq 2(n+1) \implies n \geq 5.$$

We verify that $n = 5$ works: if the initial readings are $0, 0, 0, 0, -9$, then adding a reading of 3 gives an average of $\frac{0+0+0+0+(-9)+3}{6} = -1$, and adding a reading of 15 gives an average of $\frac{0+0+0+0+(-9)+15}{6} = 1$. Therefore, the smallest possible number of readings is $\boxed{5}$.

I5. Answer: $\frac{1}{3}$

Place Utkarsh at point A and his house at point B, one mile away (so $AB = 1$). He walks 1 mile in a uniformly random direction θ, ending at point C. This point could lie anywhere on the unit circle centered at A.

He is closer to his house after walking if and only if $BC < 1$, i.e., C lies strictly inside the unit circle centered at B. The boundary of this circle passes through A, and we want to find the measure of angles θ for which C is inside it.



The circle of radius 1 centered at A and the circle of radius 1 centered at B intersect at two points C_1 and C_2. Since $AB = AC_1 = AC_2 = BC_1 = BC_2 = 1$ as they are all radii, both triangles ABC_1 and ABC_2 are equilateral. Therefore the angle $\angle C_1AC_2 = 60^\circ + 60^\circ = 120^\circ$.

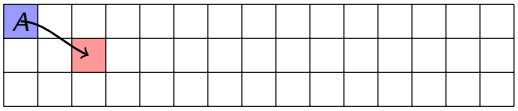
Utkarsh ends up closer to his house when his direction θ from A falls within this 120° arc (the arc from C_2 to C_1 through B). Therefore, the probability is:

$$\frac{120^\circ}{360^\circ} = \boxed{\frac{1}{3}}.$$

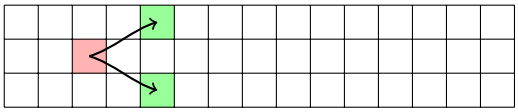
I6. Answer: 8

To go from A to B, the total horizontal displacement must be $15 - 1 = 14$. Each knight move contributes a horizontal displacement of ± 1 or ± 2, so the maximum possible horizontal displacement in 7 moves is $7 \times 2 = 14$. Since we need exactly 14, every single move must go $+2$ horizontally, which forces every move to go ± 1 vertically.

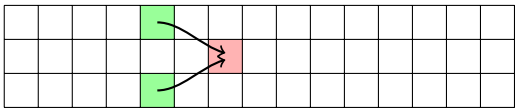
This means the first move is forced: there is only one initial move with a $+2$ horizontal displacement.



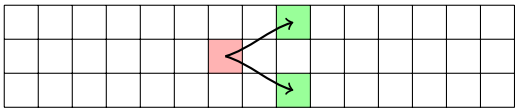
The second move has 2 choices (up or down):



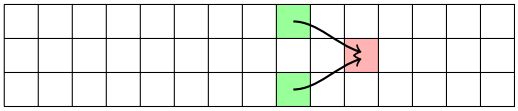
The third move is now forced, from either of these positions:



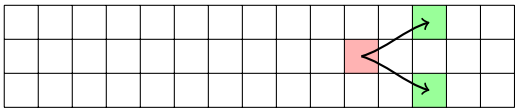
The fourth move again has 2 choices (up or down):



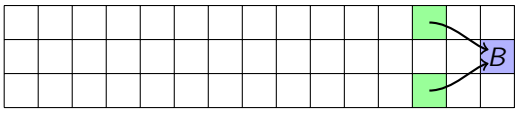
The fifth move is again forced from either of these positions:



The sixth move again has 2 choices (up or down):



Finally, the last move needs to arrive at B, which is possible from either of these positions:



Multiplying, the total number of paths is $1 \times 2 \times 1 \times 2 \times 1 \times 2 \times 1 = \boxed{8}$.

I7. Answer: 101

We have $210 = 2 \cdot 3 \cdot 5 \cdot 7$. Its 16 divisors are the 2^4 products of subsets of $\{2, 3, 5, 7\}$.

We claim the unique set of 6 divisors with no two dividing each other is:

$$\{6,\ 10,\ 14,\ 15,\ 21,\ 35\} = \{2 \cdot 3,\ 2 \cdot 5,\ 2 \cdot 7,\ 3 \cdot 5,\ 3 \cdot 7,\ 5 \cdot 7\},$$

the six products of exactly two of the four primes. First, none of these six divides another: any two of them share at most one prime factor, so neither can divide the other (a divisor would need to contain all prime factors of the dividend, but both numbers in our set have exactly 2 prime factors). Their sum is

$$6 + 10 + 14 + 15 + 21 + 35 = 101.$$

It remains to show this is the *only* valid set of 6. We verify that no other 6 divisors of 210 satisfy this property.

- **If 1 is chosen:** then 1 divides every other divisor, so no other element can be chosen. We cannot reach size 6.
- **If 210 is chosen:** every other divisor divides 210, so no other element can be chosen.
- **If a prime p (say $p = 2$) is chosen:** then we cannot choose any multiple of 2. The remaining divisors are $\{1, 3, 5, 15, 7, 21, 35, 105\}$. However, we can pick at most one number from each of the pairs $(1, 3), (5, 15), (7, 21), (35, 105)$, so we cannot reach size 6. The other primes fail similarly.
- **If a three-prime product (say $105 = 3 \cdot 5 \cdot 7$) is chosen:** then we cannot choose any divisor of 105, leaving only the multiples of 2: $\{2, 6, 10, 30, 14, 42, 70, 210\}$. However, we can pick at most one number from each of the pairs $(2, 6), (10, 30), (14, 42), (70, 210)$, so we cannot reach size 6. The other three-prime products fail similarly.

Therefore the only valid set of size 6 is $\{6, 10, 14, 15, 21, 35\}$, with sum $\boxed{101}$.

I8. Answer: 48

Construction: Consider the following distribution of the 50 students:

- 24 students rank $B \succ A \succ C$.
- 24 students rank $A \succ C \succ B$.
- 2 students rank $C \succ B \succ A$.

Check: $B \succ A$ is preferred by the 24 students in the first group, and the 2 students in the third group, giving 26 total. Similarly, $C \succ B$ is preferred by the 24 students in the second group, and the 2 in the third group, giving 26 total. So, we have that $A \succ C$ is preferred by the $24 + 24 = 48$ students in the first and second groups, giving our lower bound of 48.

Bound: Since 26 of the 50 students rank $B \succ A$, and 26 out of 50 rank $C \succ B$, then at least $26 + 26 - 50 = 2$ students have both of these rankings; that is, they rank $B \succ A$ and $C \succ B$. Therefore, by transitivity, these 2 students must rank $C \succ A$, so at most $50 - 2 = 48$ of the students rank $A \succ C$.

The construction achieves the bound, so the answer is $\boxed{48}$.

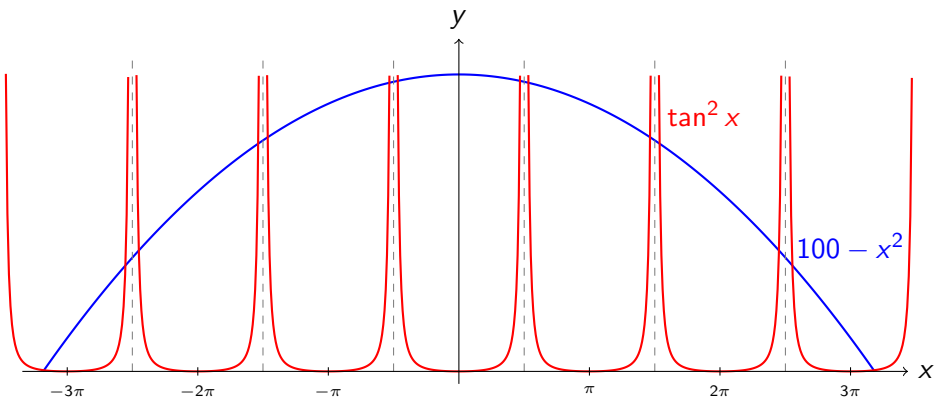
I9. Answer: 14

Rearrange the equation as $\tan^2 x = 100 - x^2$. We count intersections of the curve $y = \tan^2 x$ with the curve $y = 100 - x^2$.

The parabola $y = 100 - x^2$ is non-negative only for $x \in [-10, 10]$, while $\tan^2 x$ is always nonnegative due to the square, so all solutions lie in the interval $x \in [-10, 10]$.

The curve $y = \tan^2 x$ consists of continuous U-shaped branches on each interval between consecutive asymptotes of $\tan x$. On each branch, $\tan^2 x = 0$ at the center (a multiple of π) and $\tan^2 x \to +\infty$ at both ends. The asymptotes of $\tan x$ within $[-10, 10]$ occur at $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}$ (the next one, $\frac{7\pi}{2} > 10$, lies outside). These six asymptotes create seven branches of $y = \tan^2 x$.

We now count intersections branch by branch:

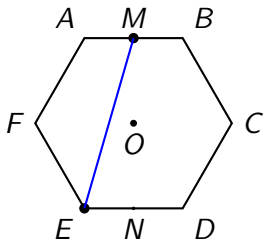


On each branch, $\tan^2 x$ starts at $+\infty$, drops to its minimum of 0 at the center (a multiple of π), then rises back to $+\infty$. The parabola $100 - x^2$ stays continuous and bounded within each branch, and is positive at each branch's center (note that $3\pi \approx 9.42 < 10$). Therefore, each branch gives us 2 intersections, for a total of $2 \times 7 = \boxed{14}$ solutions.

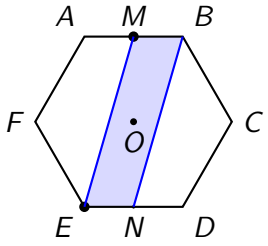
I10. Answer: $\dfrac{2}{\sqrt{13}}$

Let the regular hexagon be $ABCDEF$ with center O and side length s. We want the minimum s such that every point in the hexagon can be covered by some unit segment lying entirely within the hexagon.

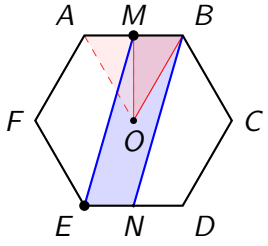
Let M be the midpoint of side AB. Notice that the longest segment through M that stays inside the hexagon is the segment connecting M to one of its opposite vertices E (or D).



Therefore, to cover M, this segment must have length at least 1. We now show that a length-1 segment here suffices. By sweeping this segment rightwards, it can be seen that any point in parallelogram $MENB$ can be covered (where N is the midpoint of side ED:

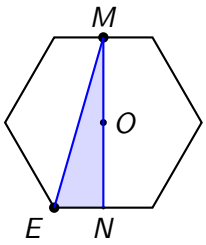


Notice that this parallelogram contains triangle MBO, so any point in triangle MBO can be covered by a length-1 segment. Reflecting over line MN, we also have that any point in triangle AMO can be covered, so together this covers the entirety of triangle ABO.



Finally, rotating through by $60^\circ, 120^\circ, 180^\circ, 240^\circ$, and 300° shows that the entire hexagon can be covered.

It remains to compute the side length of the hexagon, given that $EM = 1$. Let $EN = x$; then since ENO is a 30-60-90 triangle we have $NO = \sqrt{3}x$, and so $NM = \sqrt{12}x$.



Using the Pythagorean Theorem on triangle ENM, we get:

$$\begin{aligned} EN^2 + NM^2 &= EM^2 \\ \implies x^2 + (\sqrt{12}x)^2 &= 1 \\ \implies x^2 + 12x^2 &= 1 \\ \implies 13x^2 &= 1 \\ \implies x^2 &= \frac{1}{13} \\ \implies x &= \frac{1}{\sqrt{13}} \end{aligned}$$

.

Finally, the side length is twice EN, or $2x$, giving the answer of $\boxed{\frac{2}{\sqrt{13}}}$.

I11. Answer: 154

The runners are at the diametrically opposite point when each has completed a half-integer number of laps. If runner i has lap time ℓ_i minutes and they all meet at the opposite point at time t, then t/ℓ_i must be a positive half-integer (i.e., an odd multiple of $1/2$) for each i.

Let p, q, r be the number of half-laps that runners 1, 2, and 3 completed respectively. Then, we have $t = p\frac{\ell_1}{2} = q\frac{\ell_2}{2} = r\frac{\ell_3}{2}$ for odd positive integers p, q, r. This is equivalent to the condition $p\ell_1 = q\ell_2 = r\ell_3$.

For odd integers p and q to satisfy $p\ell_1 = q\ell_2$, write $\ell_1 = 2^a m$ and $\ell_2 = 2^b n$ with m, n odd. Then $p \cdot 2^a m = q \cdot 2^b n$. Since p and q are odd, both sides have exactly 2^a and 2^b as their 2-part respectively, so we need $a = b$. In other words, ℓ_1 and ℓ_2 must have the *same* highest power of 2 dividing them.

Applying this to all three pairs, the condition for such a time t to exist is:

$$v_2(x) = v_2(y) = v_2(z),$$

where $v_2(n)$ denotes the largest power of 2 dividing n.

We count triples (x, y, z) with $1 \le x, y, z \le 10$ satisfying this. The values 1 through 10 grouped by 2-adic valuation are:

- $v_2 = 0$ (odd numbers): $\{1, 3, 5, 7, 9\}$ — **5 values**.
- $v_2 = 1$ (divisible by 2 but not 4): $\{2, 6, 10\}$ — **3 values**.
- $v_2 = 2$ (divisible by 4 but not 8): $\{4\}$ — **1 value**.
- $v_2 = 3$ (divisible by 8 but not 16): $\{8\}$ — **1 value**.

Therefore, the number of valid triples is:

$$5^3 + 3^3 + 1^3 + 1^3 = 125 + 27 + 1 + 1 = \boxed{154}.$$

I12. Answer: $\dfrac{2027}{6075}$

Stone 1 is never eliminated (it always beats any other stone), so it remains until the end. Stone 2 is eliminated if and only if it is ever paired with stone 1 (which immediately eliminates stone 2). Thus stone 2 survives to the final pair if and only if it is never drawn together with stone 1.

When there are k stones remaining, there are $\binom{k}{2}$ equally likely pairs. The probability the pair $\{1,2\}$ is chosen at this step is $\dfrac{1}{\binom{k}{2}} = \dfrac{2}{k(k-1)}$.

Stone 2 survives all steps $k = 2026, 2025, \ldots, 3$ (the process stops when $k = 2$, at which point stones 1 and 2 are guaranteed to both be present). The probability of survival is:

$$p = \prod_{k=3}^{2026} \left(1 - \frac{2}{k(k-1)}\right) = \prod_{k=3}^{2026} \frac{k(k-1)-2}{k(k-1)} = \prod_{k=3}^{2026} \frac{(k-2)(k+1)}{k(k-1)}.$$

We separate this into two telescoping products:

$$p = \prod_{k=3}^{2026} \frac{k-2}{k-1} \cdot \prod_{k=3}^{2026} \frac{k+1}{k}.$$

First product:

$$\prod_{k=3}^{2026} \frac{k-2}{k-1} = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdots \frac{2024}{2025} = \frac{1}{2025}.$$

Second product:

$$\prod_{k=3}^{2026} \frac{k+1}{k} = \frac{4}{3} \cdot \frac{5}{4} \cdot \frac{6}{5} \cdots \frac{2027}{2026} = \frac{2027}{3}.$$

Therefore:

$$p = \frac{1}{2025} \cdot \frac{2027}{3} = \frac{2027}{6075}.$$

The answer is $\boxed{\dfrac{2027}{6075}}$.

I13. Answer: 961

Suppose $x^6 + n = P(x)Q(x)R(x)$, where P, Q, R are quadratics with integer coefficients. Then the six roots of $x^6 + n$ must be partitioned into three pairs, so that each pair corresponds to the roots of one of the quadratics. We compute these roots, then verify whether any possible pairings allow all three quadratics to have integer coefficients.

Case 1: $n > 0$**.**

We find the solutions to $x^6 + n = 0$:

$$x^6 + n = 0 \implies x^6 = -n = e^{i\pi} n \implies x = e^{i\pi/6} n^{1/6} (e^{2i\pi k/6}), \qquad k = 0, 1, 2, 3, 4, 5.$$

Thus they lie on the circle of radius $n^{1/6}$ at arguments

$$\frac{\pi}{6}, \frac{3\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{9\pi}{6}, \frac{11\pi}{6}.$$

Writing them explicitly, the roots are:

$$n^{1/6}\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right),\quad n^{1/6}i,\quad n^{1/6}\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right),$$

$$n^{1/6}\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}i\right),\quad -n^{1/6}i,\quad n^{1/6}\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right).$$

Each of these roots are nonreal, so they must be paired with their complex conjugates (because the sum of the roots becomes the negative linear term of the quadratic, which must be real). This forces the following pairs:

$$n^{1/6}\left(\frac{\sqrt{3}}{2}\pm\frac{1}{2}i\right),\quad n^{1/6}\left(-\frac{\sqrt{3}}{2}\pm\frac{1}{2}i\right),\quad \pm n^{1/6}i$$

The corresponding quadratics are:

$$x^2-\sqrt{3}\,n^{1/6}x+n^{1/3},\qquad x^2+\sqrt{3}\,n^{1/6}x+n^{1/3},\qquad x^2+n^{1/3}.$$

For these to have integer coefficients we must have $n^{1/3}\in\mathbb{Z}$ and $\sqrt{3}\,n^{1/6}\in\mathbb{Z}$.. The second condition forces $n^{1/6}=\sqrt{3}k$ for some integer k, and so $n=(\sqrt{3}k)^6=27k^6$. Since $n\le 2026$, only $k=1,2$ work, giving us the solutions $n=27,1728$, which we can verify also satisfy the first condition $n^{1/3}\in\mathbb{Z}$.

Case 2: $n<0$**.**

Write $n=-m$ with $m>0$. Then the equation becomes $x^6=m$, whose roots are:

$$m^{1/6}e^{2ik\pi/6},\qquad k=0,1,\ldots,5.$$

Writing them explicitly, these are:

$$\pm m^{1/6},\quad m^{1/6}\left(\tfrac{1}{2}\pm\tfrac{\sqrt{3}}{2}i\right),\quad m^{1/6}\left(-\tfrac{1}{2}\pm\tfrac{\sqrt{3}}{2}i\right).$$

The last two pairs of these roots are conjugate pairs, so they are forced together. This forces the first pair of roots together as well, giving us the following quadratics:

$$x^2-m^{1/3},\quad x^2-m^{1/6}x+m^{1/3},\quad x^2+m^{1/6}x+m^{1/3}.$$

Thus we must have $m^{1/3}\in\mathbb{Z},\quad m^{1/6}\in\mathbb{Z}$.

The first condition is superfluous given the second one. Hence $m=t^6$ for some integer t, giving $n=-t^6$. Since $|n|\le 2026$, only $t=1,2,3$ work, giving the solutions $n=-1,-64,-729$.

Case 3: $n=0$**.** We have $x^6=(x^2)(x^2)(x^2)$, so taking $P(x)=Q(x)=R(x)=x^2$ works.

Thus, the solutions within the desired range are $n\in\{-729,-64,-1,0,27,1728\}$, which sum to $\boxed{961}$.

I14. Answer: $\dfrac{79}{2}$

We track the state of the four cups by the multiset of their water amounts (as fractions of the total, which is always 1). We reduce to four states, not including the finishing state:

- **State S_0:** $\{1, 0, 0, 0\}$, the starting state.
- **State S_1:** "bad abcc": $\{a, b, c, c\}$ where $c \neq \frac{1}{4}$ and $a \neq b$. This state is called *bad* because we cannot finish in one step.
- **State S_2:** "aabb": $\{a, a, b, b\}$ where $a \neq b$.
- **State S_3:** "good abcc": $\{a, b, \frac{1}{4}, \frac{1}{4}\}$ where $a \neq b$. This state is called *good* because we can finish in one lucky step.

Let E_0, E_1, E_2, E_3 be the expected number of steps until finished from states S_0, S_1, S_2, S_3 respectively. Our goal is to determine E_0. We compute expressions for each of these values in terms of each other:

State S_0 (start): If one of our two cups is the 1, we end up with water amounts $\{\frac{1}{2}, \frac{1}{2}, 0, 0\}$, which is the "aabb" state S_2. This happens $1/2$ of the time. The other $1/2$ of the time, where we pick two 0s, do nothing. Therefore, $E_0 = 1 + \frac{1}{2}E_0 + \frac{1}{2}E_2 \implies E_0 = 2 + E_2$.

State S_1 (bad abcc): Suppose the multiset of water amounts is $\{a, b, c, c\}$. Then:

- If we choose the two c's, nothing happens, and we stay in state S_1. This happens $1/6$ of the time.
- If we choose the a and the b, then the multiset of water amounts becomes $\{\frac{a+b}{2}, \frac{a+b}{2}, c, c\}$. Since $c \neq \frac{1}{4}$, we're not finished, and we land in the "aabb" state S_2. This happens $1/6$ of the time.
- If we choose the a and one of the c's, the multiset of water amounts becomes $\{\frac{a+c}{2}, b, \frac{a+c}{2}, c\}$. We need to check if this is good or bad abcc. We know that $a \neq b$, so $(a + c) \neq (b + c)$. Then using the sum of all cups $(a + c) + (b + c) = 1$, this means $(a + c) \neq \frac{1}{2}$. Therefore $\frac{a+c}{2} \neq \frac{1}{4}$ and so this is "bad abcc", state S_1. This happens $1/3$ of the time.
- If we choose the b and a c, we similarly stay in "bad abcc", state S_1. This also happens $1/3$ of the time.

Therefore, $E_1 = 1 + \frac{1}{6}E_2 + \frac{5}{6}E_1 \implies E_1 = 6 + E_2$.

State S_2 (aabb): Suppose the multiset of water amounts is $\{a, a, b, b\}$. Note that the sum of all cups $a + a + b + b = 1$, and so $a + b = \frac{1}{2}$.

- If we pick the two a's or the two b's, nothing happens, and we stay in state S_2. This happens $1/3$ of the time.
- Otherwise, the multiset of water amounts becomes $\{a, \frac{a+b}{2}, \frac{a+b}{2}, b\}$. Since $a + b = \frac{1}{2}$, we have $\frac{a+b}{2} = \frac{1}{4}$, and so this is "good abcc", state S_3. This happens $2/3$ of the time.

Therefore, $E_2 = 1 + \frac{1}{3}E_2 + \frac{2}{3}E_3 \implies E_2 = \frac{3}{2} + E_3$.

State S_3 (good abcc): Suppose the multiset of water amounts is $\{a, b, c, c\}$. Then:

- If we choose the two c's, nothing happens, and we stay in state S_3. This happens $1/6$ of the time.
- If we choose the a and the b, then the multiset of water amounts becomes $\{\frac{a+b}{2}, \frac{a+b}{2}, c, c\}$. Since $c = \frac{1}{4}$, we're finished. This happens $1/6$ of the time.

- If we choose the a and one of the c's, the multiset of water amounts becomes $\{\frac{a+c}{2}, b, \frac{a+c}{2}, c\}$. We need to check if this is good or bad abcc. It's the same process as before: we know that $a \neq b$, so $(a+c) \neq (b+c)$. Then using the sum of all cups $(a+c)+(b+c)=1$, this means $(a+c) \neq \frac{1}{2}$. Therefore $\frac{a+c}{2} \neq \frac{1}{4}$ and so this is "bad abcc", state S_1. This happens $1/3$ of the time.
- If we choose the b and a c, we similarly stay in "bad abcc", state S_1. This also happens $1/3$ of the time.

Therefore, $E_3 = 1 + \frac{1}{6}E_3 + \frac{2}{3}E_1$.

Substituting in $E_1 = 6 + E_2$, we get: $E_3 = 1 + \frac{1}{6}E_3 + \frac{2}{3}(6 + E_2) = 5 + \frac{1}{6}E_3 + \frac{2}{3}E_2$.

Substituting in $E_2 = \frac{3}{2} + E_3$, we get: $E_3 = 5 + \frac{1}{6}E_3 + \frac{2}{3}(\frac{3}{2} + E_3) = 6 + \frac{5}{6}E_3$.

Solving, we get $E_3 = 36$. So $E_2 = \frac{3}{2} + E_3 = \frac{75}{2}$, and so $E_0 = 2 + E_2 = \boxed{\frac{79}{2}}$.

I15. Answer: 240

A function $f(x)$ is hypercubic if it has the form $f(x) = ((\cdots((x+a_1)^3+a_2)^3+\cdots)^3+a_{n-1})^3+a_n$ for integers a_i and some $n \geq 1$.

Two hypercubic functions f_i, f_j cannot both belong to S if $f_i(m) \equiv f_j(m) \pmod{10}$ for all integers m. So $|S|$ equals the number of distinct functions mod 10 that hypercubic functions can have, i.e., the number of distinct functions $\{0,1,\ldots,9\} \to \{0,1,\ldots,9\}$ (taking inputs and outputs mod 10) that arise from hypercubic functions.

Since $10 = 2 \times 5$, knowing a value mod 10 is the same as knowing it mod 2 and mod 5 separately. So the number of distinct functions mod 10 equals the number of distinct functions mod 2 times the number mod 5.

We will first work mod 5 (the mod 2 case is handled separately below). Let T_a denote the shift $x \mapsto x + a \pmod 5$ and C denote the cube $x \mapsto x^3 \pmod 5$. We may simplify any composition of these operations to alternate between T's and C's:

- Two consecutive shifts T_a followed by T_b compose to a single shift T_{a+b}, so we can merge adjacent shifts.
- We can always insert T_0 (the identity) between two consecutive cubes, or at the start/end, without changing the function. So we may assume the form $T_{a_n} \circ C \circ T_{a_{n-1}} \circ C \circ \cdots \circ C \circ T_{a_1}$: alternating shifts and cubes, starting and ending with a shift.

So every composition of T_a's and C's can be reduced to a hypercubic function (i.e., a composition of these functions in alternating order, starting and ending with a shift).

What does C do mod 5? Computing $x^3 \pmod 5$ for each residue:

$$0^3 = 0, \quad 1^3 = 1, \quad 2^3 = 3, \quad 3^3 = 2, \quad 4^3 = 4.$$

So C fixes 0, 1, and 4, and swaps $2 \leftrightarrow 3$. In particular, C is a bijection from $\{0,1,2,3,4\}$ to itself. Similarly, each T_a is a bijection. Since compositions of bijections are bijections, every hypercubic function is a bijection mod 5 — i.e., a *permutation* of $\{0,1,2,3,4\}$.

There are $5! = 120$ permutations of $\{0,1,2,3,4\}$ in total. We will show that *every* permutation is achievable.

Building a swap of any two elements. The function C swaps 2 and 3 (fixing everything else). More generally, we can produce a function that swaps any two elements $a+2$ and $a+3$ (consecutive mod 5), while fixing all others, as follows. Consider:

$$f(x) = (x-a)^3 + a \pmod 5.$$

This is the hypercubic function $T_a \circ C \circ T_{-a}$ (shift by $-a$, cube, shift back by a). For each residue x:

$$\begin{aligned} x = a &\mapsto 0^3 + a = a, \\ x = a+1 &\mapsto 1^3 + a = a+1, \\ x = a+2 &\mapsto 2^3 + a = 3 + a \equiv a+3, \\ x = a+3 &\mapsto 3^3 + a = 2 + a \equiv a+2, \quad \text{(swapped!)} \\ x = a+4 &\mapsto 4^3 + a = 4 + a \equiv a+4. \end{aligned}$$

By choosing a appropriately, we can swap any pair of elements that differ by 1 in this "shifted" sense. Concretely:

- $a = 0$: swaps $2 \leftrightarrow 3$.
- $a = 1$: swaps $3 \leftrightarrow 4$.
- $a = 2$: swaps $4 \leftrightarrow 0$.
- $a = 3$: swaps $0 \leftrightarrow 1$.
- $a = 4$: swaps $1 \leftrightarrow 2$.

So we can swap any two *adjacent* elements of $\{0,1,2,3,4\}$ (where adjacency wraps around mod 5).

Every permutation is achievable. We now show that by composing these adjacent swaps and shifts, we can produce any desired permutation σ of $\{0,1,2,3,4\}$. We build σ step by step:

For each $k = 0,1,2,3$, we bring $\sigma(k)$ to position k by a sequence of adjacent swaps: if $\sigma(k)$ currently sits at position $j > k$, we successively swap it leftward: swap positions j and $j-1$, then $j-1$ and $j-2$, and so on until it reaches position k. Each such swap is an operation of the form $T_a \circ C \circ T_{-a}$ as constructed above, which fixes all elements not involved in the swap. Since all swaps involve positions $\geq k$, the values at positions $0, \ldots, k-1$ (already correctly set) are not disturbed. Once positions $0,1,2,3$ are correct, position 4 is automatically correct since it is the only remaining value.

Therefore every permutation of $\{0,1,2,3,4\}$ is achievable, giving $5! = 120$ distinct functions mod 5.

Mod 2. Since $x^3 \equiv x \pmod 2$ (as $0^3 = 0$ and $1^3 = 1$), the cube operation does nothing mod 2. So every hypercubic function satisfies $f(x) \equiv x + c \pmod 2$ for some constant c, giving exactly 2 distinct functions mod 2 (namely $f \equiv x$ and $f \equiv x+1$).

Conclusion. The total number of distinct functions mod 10 is $2 \times 120 = \boxed{240}$.